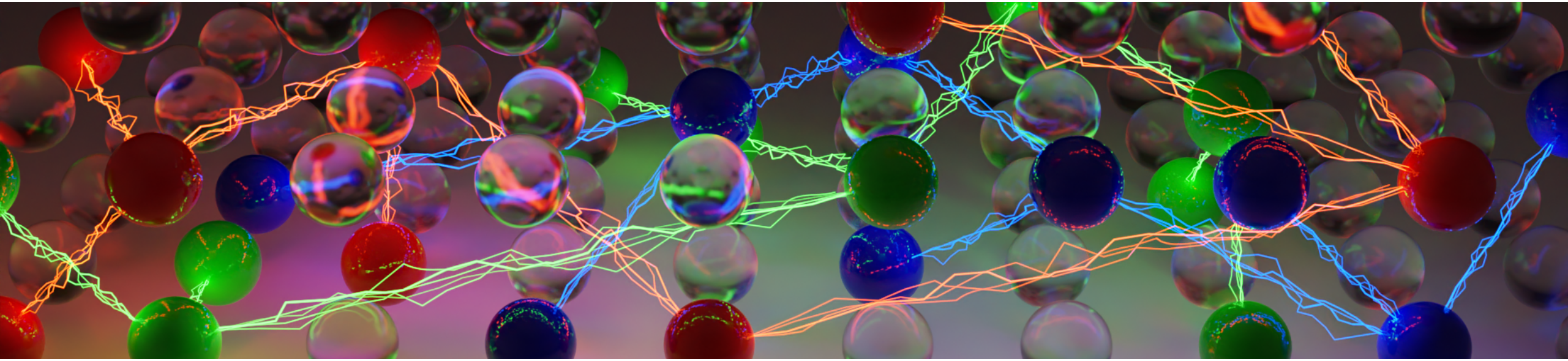
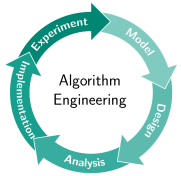


Practical SAT Solving



Lecture 3 – Elementary SAT Solving Algorithms

Markus Iser, Dominik Schreiber | May 5, 2025

SAtRes
Scalable Automated Reasoning

Overview

Recap. Lecture 2

- Tractable Subclasses
- Constraint Encodings
- Encoding Techniques

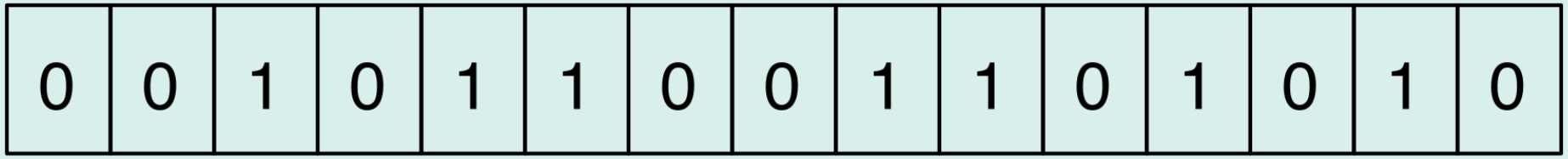
Today's Topics: Elementary SAT Algorithms

- Local Search
- Resolution
- DP Algorithm
- DPLL Algorithm

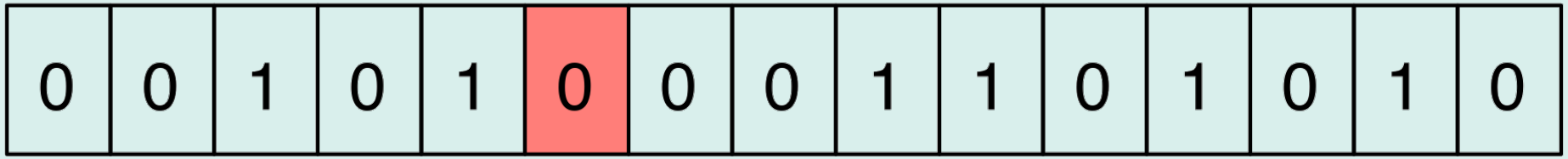
Stochastic Local Search (SLS)

Minimize the Number of Unsatisfied Clauses

Start with a random complete variable assignment α:



Repeatedly flip variables in α to decrease the number of unsatisfied clauses:



Stochastic Local Search (SLS)

Properties of SLS Algorithms

Local search algorithms are incomplete: They cannot show unsatisfiability!

Challenges:

- Which variable should be flipped next?

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  - select variable from an unsatisfied clause
  - select variable that maximizes the number of satisfied clauses
- How to avoid getting stuck in local minima?

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Properties of SLS Algorithms

Local search algorithms are incomplete: They cannot show unsatisfiability!

Challenges:

- Which variable should be flipped next?
  - select variable from an unsatisfied clause
  - select variable that maximizes the number of satisfied clauses
- How to avoid getting stuck in local minima?
  - Randomization!

Classic SLS Algorithms

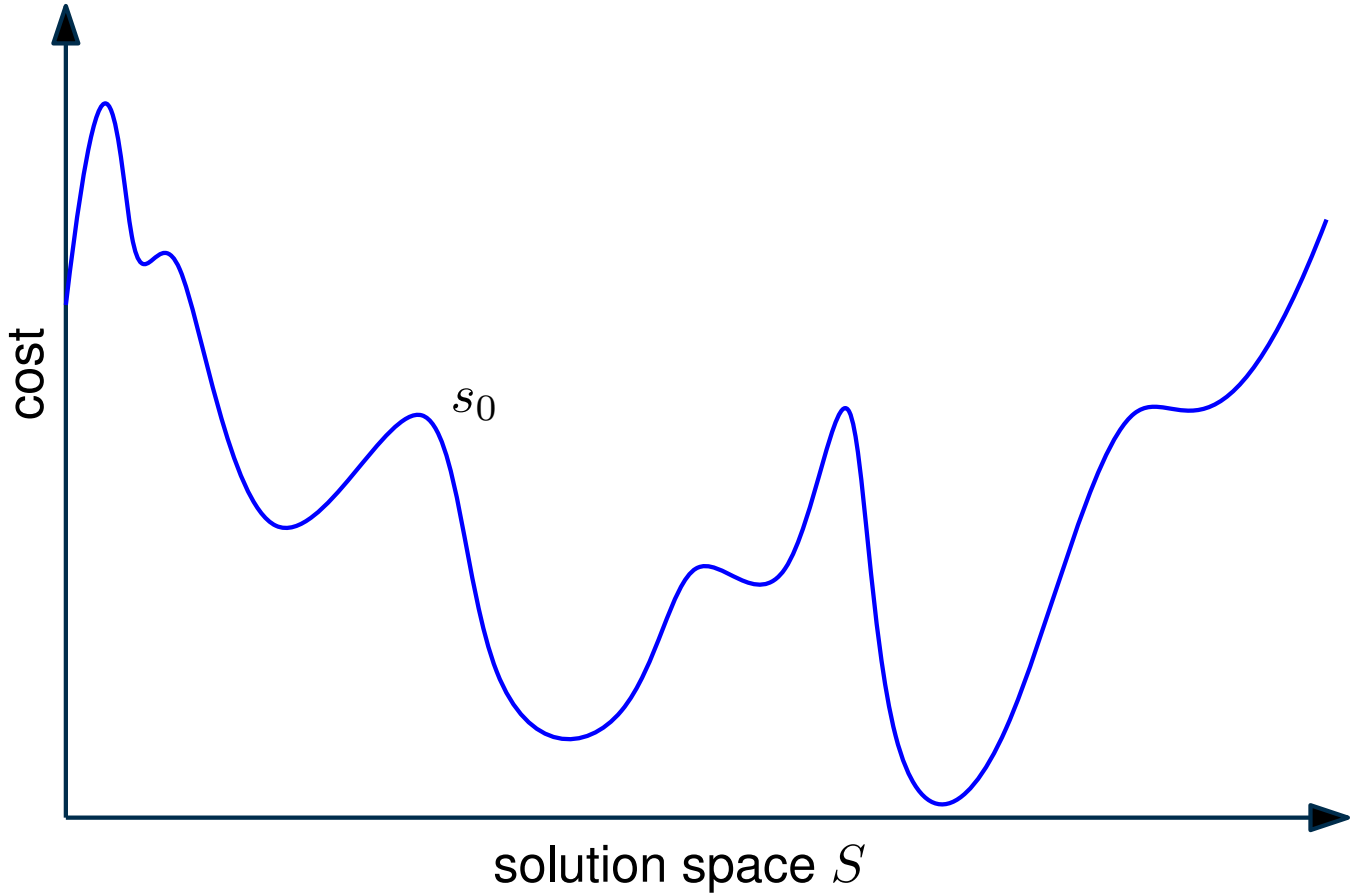
GSAT

Greedy local search algorithm

Algorithm: GSAT
Input: ClauseSet S
Output: Assignment α, or Nothing

```
1 for i = 1 to MAX_TRIES do
2 |  α = random-assignment to variables in S
3 |  for j = 1 to MAX_FLIPS do
4 |  |  if α satisfies all clauses in S then return α
5 |  |  x = variable that produces least number of unsatisfied clauses
  |  |   when flipped
6 |  |  flip x
7 return Nothing                                  // no solution found
```

SLS: Local Minima



[Illustration Adapted from: Alan Mackworth, UBC, Canada]

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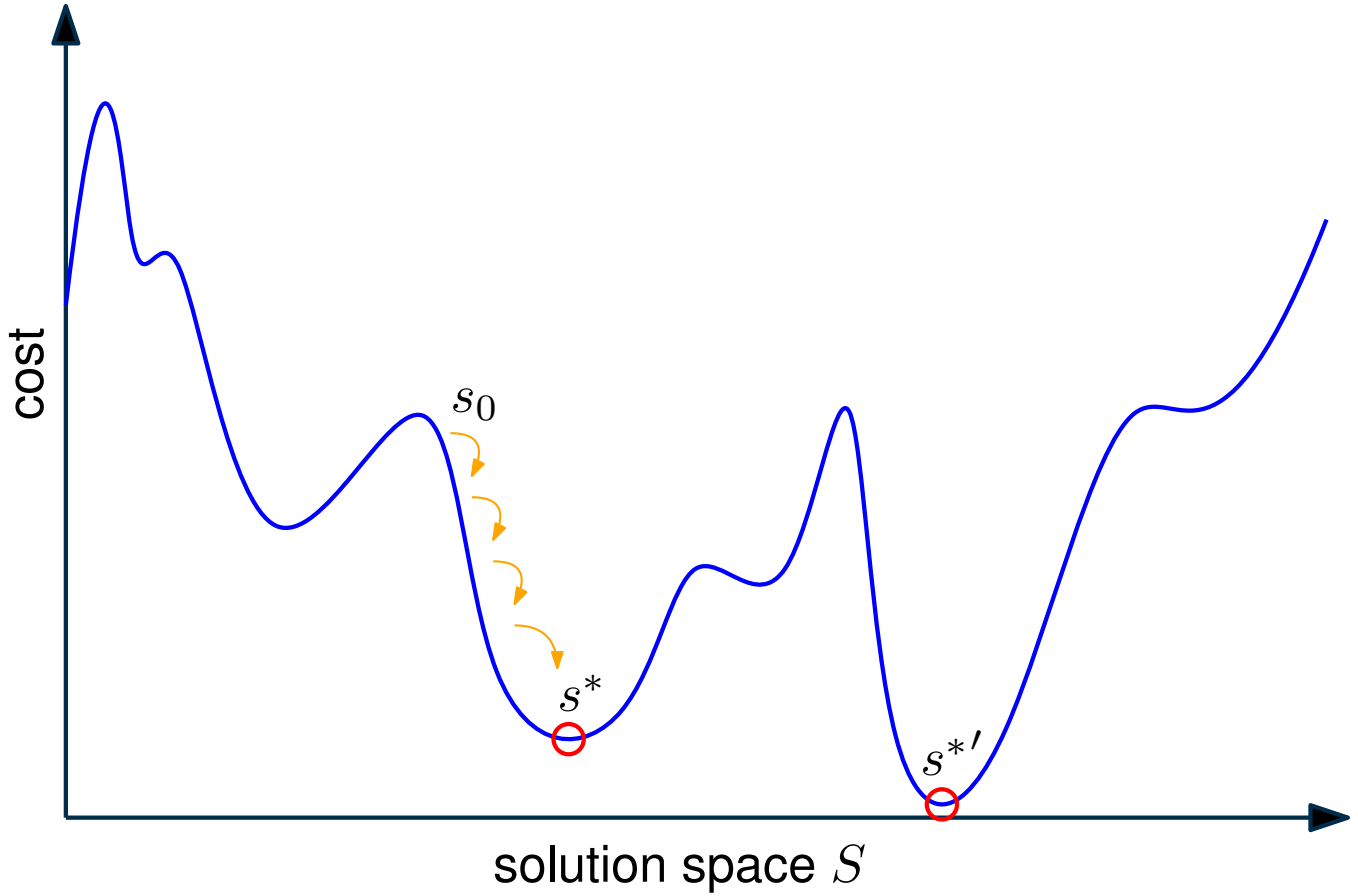

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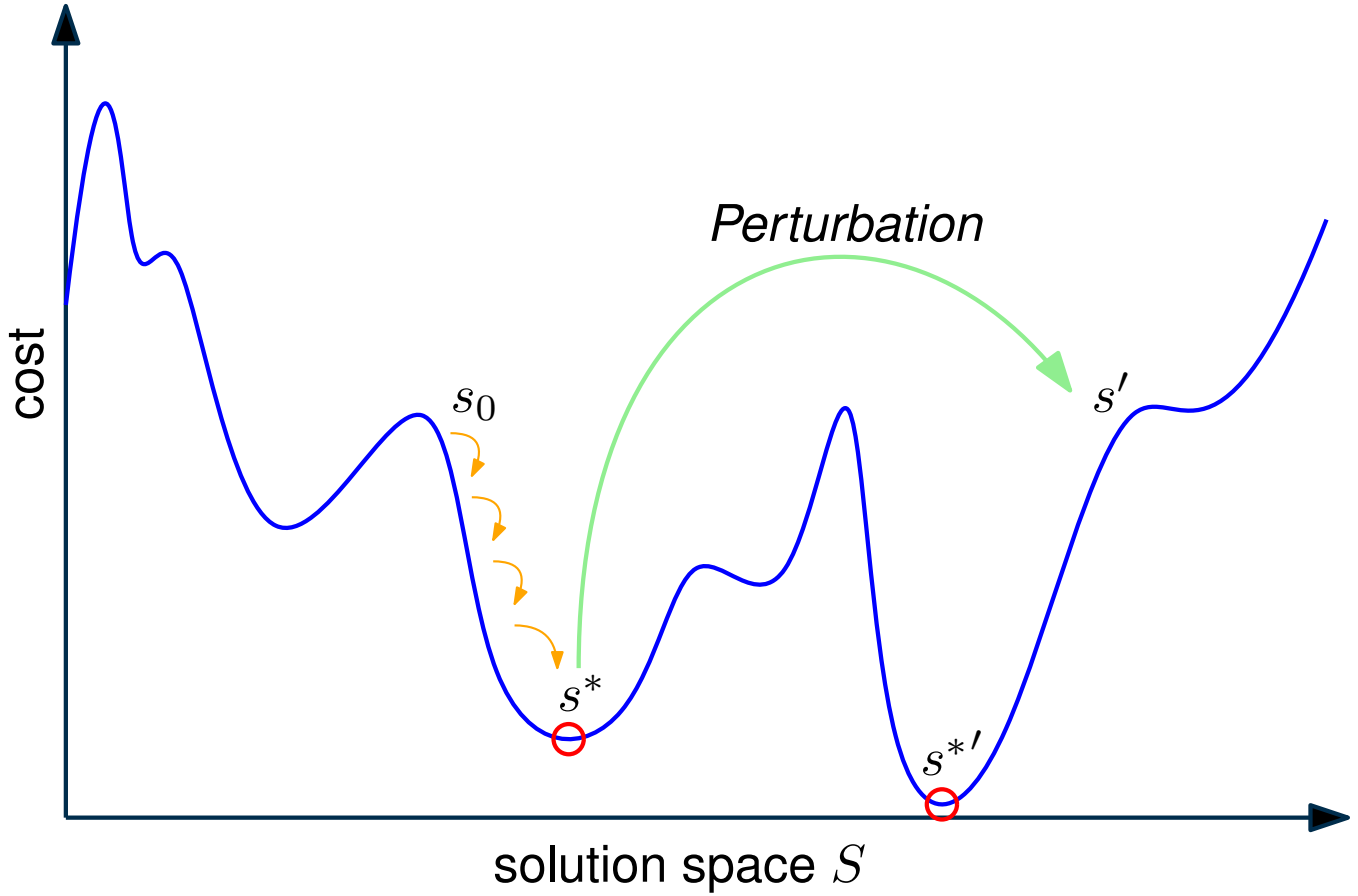

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Classic SLS Algorithms

WalkSAT

Variant of GSAT

Try to avoid local minima by introducing random noise.

Algorithm: WalkSAT(S)

```
1 for i = 1 to MAX_TRIES do
2     α = random-assignment to variables in S
3     for j = 1 to MAX_FLIPS do
4         if α satisfies all clauses in S then return α
5         C = random unsatisfied clause in S
6         if by flipping an x ∈ C no new unsatisfied clauses emerges
           then flip x
7         else with probability p flip an x ∈ C at random
8         otherwise, flip a variable that changes the least number of
           clauses from satisfied to unsatisfied
9 return Nothing                                   // no solution found
```

SLS: Important Notions

Consider a flip taking α to α'

breakcount number of clauses satisfied in α, but not satisfied in α'

makecount number of clauses not satisfied in α, but satisfied in α'

diffscore # unsatisfied clauses in α − # unsatisfied clauses in α'

Typically, breakcount, makecount, and/or diffscore are used to select the variable to flip.

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Recap using new nomenclature

GSAT select variable with highest diffscore

WalkSAT select variable with minimal breakcount

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Legacy of SLS

- Extremely successful and popular in early days of SAT
  - SLS outperformed early resolution-based solvers, e.g., based on DP or DPLL
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  - Faster, more reliable, and complete!

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- Today, sophisticated resolution-based systematic search solvers dominate in most practical applications
  - Faster, more reliable, and complete!
- Still useful as a component in more complex solvers
  - Part of (parallel) algorithm portfolios
  - Control branching heuristics in complete search algorithms
  - Detection of autarkies in formula simplification algorithms
  - In combination with complete solvers for optimization problems (e.g., MaxSAT)

Recap

Elementary Algorithms

- Local Search
  - Examples: GSAT, WalkSAT
  - Terminology: breakcount, makecount, diffscore

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Next Up

- Resolution

Resolution

The Resolution Rule

$$\frac{P_1 \cup \{x\} \qquad P_2 \cup \{\neg x\}}{P_1 \cup P_2}$$

Resolution is a logical inference rule, which infers a conclusion (resolvent) from given premises (input clauses).

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Example (Resolution)

$$\{x_1, x_3, \neg x_7\}, \{\neg x_1, x_2\} \vdash \{x_3, \neg x_7, x_2\}$$

$$\{x_4, x_5\}, \{\neg x_5\} \vdash \{x_4\} \quad \text{(Fact)}$$

$$\{x_1, x_2\}, \{\neg x_1, \neg x_2\} \vdash \{x_1, \neg x_1\} \quad \text{(Tautological Resolvent)}$$

$$\{x_1\}, \{\neg x_1\} \vdash \{\} \quad \text{(Empty Clause)}$$

Resolution

Theorem: Resolution is sound

Given a CNF formula F with two resolvable clauses $C_1, C_2 \subseteq F$ with resolvent $\mathrm{R}(C_1, C_2)$, the following holds:

$$F \quad \equiv \quad F \wedge \mathrm{R}(C_1, C_2)$$

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Proof

Let $C_1 := \{x\} \cup P_1$ and $C_2 := \{\neg x\} \cup P_2$ such that $\mathrm{R}(C_1, C_2) = P_1 \cup P_2 =: D$.

Soundness: $F \vdash F \wedge D \implies F \models F \wedge D$

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Any satisfying assignment ϕ of F satisfies both C_1 and C_2. If ϕ satisfies x, then it satisfies some literal in P_2. Otherwise, ϕ satisfies $\neg x$ and thus satisfies some literal in P_1. As such, ϕ also satisfies D.

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Equivalence: $F \vdash F \wedge D \implies F \wedge D \models F$

Since D does not introduce new variables, $F \wedge D$ can not have more satisfying assignments than F.

Resolution

Resolution is sound and Refutation-complete

- If we manage to infer the empty clause from a CNF formula F, then F is unsatisfiable. (sound)
- If F is unsatisfiable, then there exists a refutation by resolution. (refutation-complete)
- Not all possible consequences of F can be derived by resolution. (“only” refutation-complete)

Resolution Proof

A resolution proof for F is a sequence of clauses $\langle C_1, C_2, \ldots, C_{k-1}, C_k = \emptyset \rangle$
where each C_i is either an original clause of F or a resolvent of two earlier clauses.

Example (Resolution Proof)

$$F = \{x_1, x_2\}, \{\neg x_1, x_2\}, \{x_1, \neg x_2\}, \{\neg x_1, \neg x_2\}$$

(Formula)
(Refutation)

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Example (Resolution Proof)

$$
\begin{aligned}
F &= \{x_1, x_2\}, \{\neg x_1, x_2\}, \{x_1, \neg x_2\}, \{\neg x_1, \neg x_2\} && \text{(Formula)} \\
&\equiv \{x_1, x_2\}, \{\neg x_1, x_2\}, \{x_1, \neg x_2\}, \{\neg x_1, \neg x_2\}, \{x_2\}, \{\neg x_2\}, \{\} && \text{(Refutation)}
\end{aligned}
$$

Saturation Algorithm

Algorithm: Saturation Algorithm

Input: CNF formula F

Output: {SAT, UNSAT}

```
1 while true do
2 |   R := resolveAll(F)
3 |   if R ∩ F ≠ R then F := F ∪ R
4 |   else break
5 if ⊥ ∈ F then return UNSAT
6 else return SAT
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Properties

- sound and complete – always terminates and answers correctly
- exponential time and space complexity

Unit Propagation

Unit Resolution

Resolution where at least one of the resolved clauses is a unit clause, i.e., has size one.

Example (Unit Resolution)

$$\mathrm{R}((x_1 \vee x_7 \vee \neg x_2 \vee x_4), (x_2)) = (x_1 \vee x_7 \vee x_4)$$

Unit Propagation

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Unit Propagation

Apply unit resolution until fixpoint is reached.

Example (Unit Propagation)

Usually, we are only interested in the inferred facts (unit clauses) and conflicts (empty clauses).

$$\{x_1, x_2, x_3\}, \{x_1, \neg x_2\}, \{\neg x_1\} \quad \vdash_1 \quad \{\neg x_2\}, \{x_3\}$$

Recap

Elementary Algorithms

- Local Search
  - Examples: GSAT, WalkSAT
  - Terminology: breakcount, makecount, diffscore
- Resolution
  - Soundness and Completeness
  - Saturation Algorithm (Exponential Complexity)
  - Unit Propagation

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Next Up

Improving upon saturation-based resolution: Davis Putnam (DP) Algorithm

Davis-Putnam Algorithm (Davis & Putnam, 1960)

Presented in 1960 as a SAT procedure for first-order logic.

Deduction Rules of DP Algorithm

- **Unit Resolution:** If there is a unit clause $C = \{\overline{x}\} \in F$, simplify all other clauses containing x
- **Pure Literal Elimination:** If a literal x never occurs negated in F, add clause $\{x\}$ to F
- **Case Splitting:** Put F in the form $(A \vee x) \wedge (B \vee \neg x) \wedge R$, where A, B, and R are clause sets free of x. Replace F by the clausification of $(A \vee B) \wedge R$

Apply above deduction rules (prioritizing rules 1 and 2) until one of the following situations occurs:

- $F = \emptyset \quad \rightarrow$ SAT
- $\emptyset \in F \quad \rightarrow$ UNSAT

Davis-Putnam Algorithm

Example (DP Algorithm)

$$F = \{\{x, y, \neg z, u\}, \{\neg x, y, u\}, \{x, \neg y, \neg z\}, \{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}, \{\neg x, \neg y, u\}\}$$ (Split by x)

Davis-Putnam Algorithm

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Davis-Putnam Algorithm

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Davis-Putnam Algorithm

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$F_4 = \{\{v\}, \{\neg v\}\} \vdash_1 \{\emptyset\}$ (Unit Resolution)

DP Variant: Bucket Elimination

Bucket Elimination

- Fix order $\prec$ on variables.
- **Bucket:** set of clauses with same $\prec$-maximal variable
- **Bucket Elimination:** process buckets in decreasing $\prec$-order
  - resolve all clauses in bucket
  - put resolvents in fitting bucket

Example (Bucket Elimination)

$$F = \{(x, y, \overline{z}, u), (\overline{x}, y, u), (x, \overline{y}, \overline{z}), (z, v), (z, \overline{v}), (\overline{z}, \overline{u}), (\overline{x}, \overline{y}, u)\} \qquad (x \succ y \succ z \succ u \succ v)$$

Variable	Bucket
x	$(x, y, \overline{z}, u), (\overline{x}, y, u), (x, \overline{y}, \overline{z}), (\overline{x}, \overline{y}, u)$
y	
z	$(z, v), (z, \overline{v}), (\overline{z}, \overline{u})$
u	
v	

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v	$(v), (\overline{v})$

DP: Discussion

Excerpt from Davis' and Putnam's paper

The superiority of the present procedure over those previously available is indicated in part by the fact that a formula on which Gilmore's routine for the IBM 704 causes the machine to compute for 21 minutes without obtaining a result was worked successfully by hand computation using [DP] in 30 minutes.

- Does DP improve on saturation's average time complexity?

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- Does DP avoid saturation's exponential space complexity?

  $\Rightarrow$ no — quadratic blowup in size for eliminating one variable

DPLL Algorithm (Davis et al., 1962)

Davis Putnam Logemann Loveland (DPLL) Algorithm

- DPLL is a backtracking search over partial variable assignments.
- Case splitting over a variable x branches the search over two cases x and $\neg x$:
  resulting in the simplified formulas $F_{|x=\texttt{true}}$ and $F_{|x=\texttt{false}}$
- Simplification rules:
  - **Unit Propagation:** If $\{l\} \in F$, l must be set to `true`.
  - **Pure Literal Elimination:** If x occurs only positively (or only negatively), it may be fixed to the respective value.

DPLL Algorithm

Algorithm: DPLL(ClauseSet S)

```
1  while S contains a unit clause {L} do
2  |  delete from S all clauses containing L          // unit-subsumption
3  |  delete ¬L from all clauses in S                 // unit-resolution
4  if ∅ ∈ S then return false                         // empty clause
5  while S contains a pure literal L do
6  |  delete from S all clauses containing L  // pure literal elimination
7  if S = ∅ then return true                          // no clauses
8  choose a literal L occurring in S                  // case-splitting
9  if DPLL(S ∪ {{L}}) then return true                // first branch
10 else if DPLL(S ∪ {{¬L}}) then return true          // second branch
11 else return false
```

Start with simplifications;

recurse on subformulas obtained by case-splitting;

stop if satisfying assignment found or all branches are unsatisfiable.

DPLL Algorithm with Trail

Algorithm: trailDPLL(ClauseSet S, PartialAssignment α)

```
1  while (S, α) contains a unit clause {L} do
2  |  add {L = 1} to α                                  // Unit Propagation
3  if a literal is assigned both 0 and 1 in α then
4  |  return false                                      // Conflict
5  if all literals assigned then
6  |  return true                                       // Assignment found
7  choose a literal L not assigned in α occurring in S  // Case Splitting
8  if trailDPLL(S, α ∪ {{L = 1}}) then
9  |  return true                                       // first branch
10 else if trailDPLL(S, α ∪ {{L = 0}}) then
11 |  return true                                       // second branch
12 else return false
```

(S, α) is the clause set S as "seen" under partial assignment α

No explicit pure literal elimination (too slow for the benefit it provides)

trailDPLL() leads to efficient iterative DPLL implementation

DPLL Algorithm

Properties

- DPLL always terminates

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  - If clause set S is SAT, we eventually find a satisfying α
  - If clause set S is UNSAT, the entire space of (partial) variable assignments is searched (but variable selection still matters!)
- Space complexity: linear!
  - Systematic search avoids blowup of "unfocused" DP

Recap

Elementary Algorithms

- Local Search
  - Examples: GSAT, WalkSAT
  - Terminology: breakcount, makecount, diffscore
- Resolution
  - Soundness and Completeness
  - Saturation Algorithm (Exponential Complexity)
- DP Algorithm
  - Systematized Resolution
  - Improved Average Time Complexity
- DPLL Algorithm
  - Case Splitting and Unit Propagation
  - Linear Space Complexity