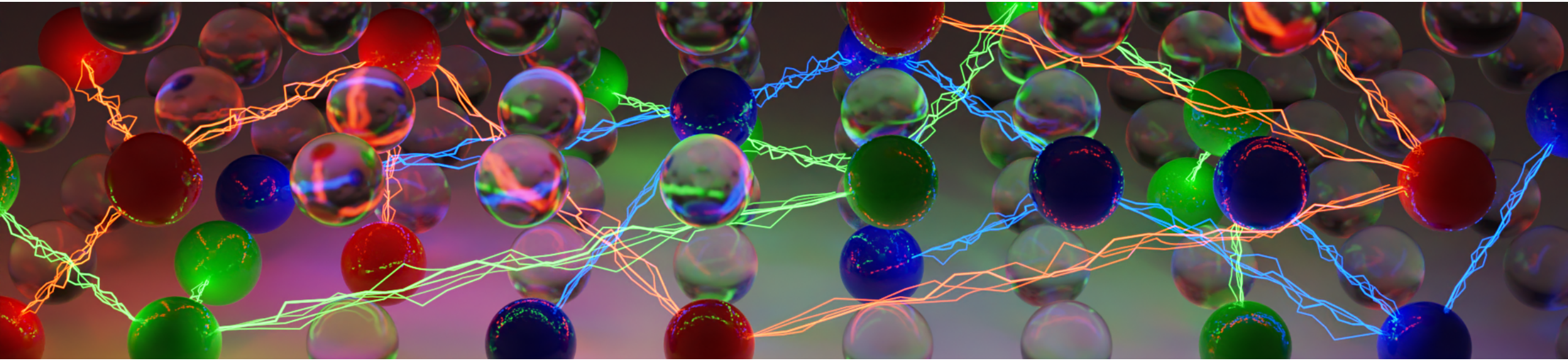
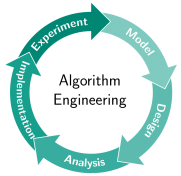


Practical SAT Solving



Lecture 9 – Proof Pragmatics & Parallel Proof Technology

Markus Iser, Dominik Schreiber | June 30, 2025

SAtRes
Scalable Automated Reasoning

Overview

Recap Lecture 8

- Popular parallelization approaches for SAT (“antisocial nerds” analogy)
  — Search space splitting, Cube&Conquer
  — Pure portfolio
  — Clause sharing portfolio
- Deep dive into Mallob system for scheduling & solving SAT tasks

Today: Proof Pragmatics & Parallel Proof Technology

- Common state-of-the-art proof formats
- Pragmatics of proof production and checking
- Producing proofs with parallel + distributed solvers
- Beyond proof files: on-the-fly checking

Proofs: Back To Basics

- Solver on unsatisfiable formula F produces sequence of clauses $P := \langle c_1, c_2, \ldots, c_n \rangle$ with $c_n = \emptyset$
- Goal: Justify for $i = 1, \ldots, n$ that $F \models c_i$, i.e., that c_i follows from F
  – actually (in practice): that $(F \cup \bigcup_{j=1}^{i-1} c_j) \models c_i$
- **Clausal Proof** $\mathcal{P}$: Expression of P with all information needed to justify all steps

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- **Clausal Proof** $\mathcal{P}$: Expression of P with all information needed to justify all steps efficiently (?)

Approach 1: Basic Clausal Proof

- **Solving**: Solver just logs each produced c_i to a file $\Rightarrow \mathcal{P} = P$
- **Checking**: Maintain clause database B initialized as $B := F$;
  for each c_i, confirm that $B \models c_i$ and then $B := B \cup c_i$

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  for each c_i, confirm that $B \models c_i$ and then $B := B \cup c_i$
- How do we perform the confirmation step?

Efficient Redundancy Checking [17]

The RUP Property

Given a clause set C and a clause c, we say that c has the Reverse Unit Propagation (RUP) property iff unit propagation on $(C \cup \{\overline{c}\})$, where $\overline{c} := \{\neg l \ : \ l \in c\}$, produces the empty clause.

- Is a clause c with RUP property w.r.t. a checker's clause set B a sound addition to B?

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  - Conflict clauses from CDCL
  - Clauses arising from many pre– and inprocessing techniques (variable elimination, subsumption, vivification, ...)
  - Actually, almost all clauses produced by out-of-the-box CaDiCaL

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  - Clauses arising from many pre– and inprocessing techniques (variable elimination, subsumption, vivification, ...)
  - Actually, almost all clauses produced by out-of-the-box CaDiCaL
- What kinds of clauses do not have the RUP property?
  - Extended Resolution steps
  - Propagation Redundancy (PR) clauses
  - ...

RUP Proof Checking

Approach 2: RUP Proof

Solving: Solver just logs each produced c_i to a file $\Rightarrow \mathcal{P} = P$.
Checking:

$B := F$
for $i = 1, \ldots, n$:
    propagate $\neg l$ in B for each $l \in c_i$
    if propagation in B does *not* yield the empty clause:
        return ERROR
    undo propagations in B
    $B := B \cup c_i$
return VALIDATED

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Checking complexity:

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Checking complexity: $\mathcal{O}(|B|)$ per step $\Rightarrow \mathcal{O}(|P|^2)$ for $|P| \gg |F|$
Checking space usage:

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Checking space usage: $\mathcal{O}(|F| + |P|)$

How to improve on both?

From RUP to DRUP [8]

Actually, a solver also deletes clauses. $\Rightarrow$ Put deletion information in the proof!

- $\mathcal{P} = (o_1, \ldots, o_N)$ where $o_i = (op_i, c_i)$
  - $op_i \in \{\texttt{add}, \texttt{delete}\}$
  - `delete`: c_i is a clause added by some o_j, $j < i$, and not `deleted` by any o_k, $j < k < i$
  - commonly uses multi-set semantics: a clause may be added (+ deleted) multiple times

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Formula:

$$
\begin{aligned}
& x_1 \vee \neg x_2 \\
\wedge\ & x_2 \vee \neg x_4 \\
\wedge\ & x_1 \vee x_2 \vee x_4 \\
\wedge\ & \neg x_1 \vee \neg x_3 \\
\wedge\ & x_1 \vee \neg x_3 \\
\wedge\ & \neg x_1 \vee x_3 \\
\wedge\ & x_1 \vee x_3 \vee \neg x_4 \\
\wedge\ & x_1 \vee x_3 \vee x_4
\end{aligned}
$$

Proof:

$$
\begin{aligned}
&\texttt{add}\ \neg x_3 \\
&\texttt{add}\ x_1 \vee x_2 \\
&\texttt{add}\ \neg x_1 \\
&\texttt{del}\ \neg x_3 \\
&\texttt{add}\ x_2 \vee x_3 \vee \neg x_4 \\
&\texttt{add}\ x_1 \vee x_2 \vee x_3 \\
&\texttt{add}\ \emptyset
\end{aligned}
$$

DRUP

Approach 3: DRUP (Deletion RUP) Proof

$B := F$
for $i = 1, \ldots, N$:
    if $op_i =$ `delete`:
        $B := B \setminus c_i$
        continue
    propagate $\neg l$ in B for each $l \in c_i$
    ... // continue as in RUP Proof

Correctness:

DRUP

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Complexity: $\mathcal{O}(|P| \times M)$ where M is the max. volume of present clauses during solving
Space usage: $\mathcal{O}(M)$ $\Rightarrow$ "fits into RAM if solving fits into RAM"

Can we further improve running time?

From DRUP to LRUP [4]

Idea: Enrich proof to accelerate unit propagation (UP) of $\overline{c_i}$ through B

- $\mathcal{P} = (o_1, \ldots, o_N)$ where $o_i = (\texttt{add}, id_i, c_i, d_i)$ **or** $o_i = (\texttt{delete}, id_i)$
  - $id_i \in \mathbb{N}^+$, $d_i = \langle d_{i1}, \ldots, d_{ik_i} \rangle$ where $d_{ij} \in \mathbb{N}^+$, $k_i \in \mathbb{N}^+$
- d_i references earlier clauses which UP needs to look at to arrive at the empty clause
  - for $1 \leq j < k_i$, clause $\# d_{ij}$ (i.e., the clause referred to by d_{ij}) must break down into a unit
  - clause $\# d_{ik_i}$ must break down into $\emptyset$

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Formula:

(1) $x_1 \vee \neg x_2$
(2) $x_2 \vee \neg x_4$
(3) $x_1 \vee x_2 \vee x_4$
(4) $\neg x_1 \vee \neg x_3$
(5) $x_1 \vee \neg x_3$
(6) $\neg x_1 \vee x_3$
(7) $x_1 \vee x_3 \vee \neg x_4$
(8) $x_1 \vee x_3 \vee x_4$

DRUP Proof:

add $\neg x_3$
add $x_1 \vee x_2$
add $\neg x_1$
del $\neg x_3$
add $x_2 \vee x_3 \vee \neg x_4$
add $x_1 \vee x_2 \vee x_3$
add $\emptyset$

LRUP Proof:

add (9) $\neg x_3$ (5, 4)
add (10) $x_1 \vee x_2$ (3, 2)
add (11) $\neg x_1$ (6, 9)
del (9)
add (12) $x_2 \vee x_3 \vee \neg x_4$ (7, 11)
add (13) $x_1 \vee x_2 \vee x_3$ (8, 12)
add (14) $\emptyset$ (11, 10, 1)

LRUP

Approach 4: LRUP (Linear RUP) Proof Checking

$B := F$
for $i = 1, \ldots, N$:
  if $op_i = \texttt{delete}$:
    $B := B \setminus \{\# id_i\}$ // delete clause referred to by id_i
    continue
  $U := \{\bar{l} \ : \ l \in c_i\}$
  for $j = 1, \ldots, k-1$:
    assert: clause $\# d_{ij}$ under U becomes a unit clause $\{u\}$ // returns **ERROR** upon failure
    $U := U \cup \{u\}$
  assert: clause $\# d_{ik_i}$ under U becomes the empty clause // returns **ERROR** upon failure
  $B := B \cup \{c_i\}$ // confirmed: $B \cup \{\overline{c_i}\} \models \emptyset$
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return VALIDATED

$\Rightarrow$ Larger proofs but much more efficient checking (often $10\times$ or more)
$\Rightarrow$ Allows for backward search from empty clause to prune all irrelevant proof lines

From (D|L)RUP to (D|L)RAT [7]

Resolution Asymmetric Tautology (recap)

Clause c has the *Resolution Asymmetric Tautology* (RAT) property in F w.r.t. literal $x \in c$
iff every resolvent $c' \in \{c \otimes_x \tilde{c} \mid \tilde{c} \in F_{\overline{x}}\}$ has the RUP property in F.

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"Only" requiring each clause $c_i \in P$ to have the RAT property (rather than RUP) allows for stronger proofs!

- For RAT clause c, $F \cup c$ is satisfiability-preserving to F but may be not equivalent to F
- Allows to express satisfiability-preserving transformations like variable addition
- As powerful as Extended Resolution

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How to incorporate RAT into proof checking?

- DRUP $\rightarrow$ **DRAT**: For each added clause c_i, find pivot literal $x \in c_i$ and confirm that c_i is RAT in B w.r.t. x
  - Convention: 1st literal of c_i must be valid pivot
  - Generate all resolvents, check RUP for every one of them

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  - Generate all resolvents, check RUP for every one of them
- LRUP $\rightarrow$ **LRAT**: Additions $(\mathsf{add}, id_i, c_i, d_i, r_i)$ with $r_i = \langle r_{i1}, \ldots, r_{im_i} \rangle$, $m_i \in \mathbb{N}_0$
  - Each r_{ij} references a clause $\tilde{c}$ and the required RUP steps for $c' = c_i \otimes_x \tilde{c}$ (like d_i for c_i in pure RUP)
  - Still need to internally maintain occurrences of each literal to check that all $\tilde{c}$ are covered

Proof (File) Formats: DRAT and LRAT

DIMACS CNF

```
p cnf 4 8
 1 -2    0
 2 -4    0
 1  2  4 0
-1 -3    0
 1 -3    0
-1  3    0
 1  3 -4 0
 1  3  4 0
```

DRAT proof

```
  -3       0
   1  2    0
  -1       0
d -3       0
   2  3 -4 0
   1  2  3 0
            0
```

LRAT proof

```
9     -3       0 5 4 0
10     1  2    0 3 2 0
11    -1       0 6 9 0
11 d   9       0
12     2  3 -4 0 7 11 0
13     1  2  3 0 8 12 0
14             0 11 10 1 0
```

These proofs only feature RUP additions. In an LRAT addition, each r_{ij} is written as the negated ID of $\tilde{c}$ followed by IDs for the RUP steps of c'.

DRAT and LRAT: Pragmatics

DRAT-based solving and checking: Common tool chain

```
./solver input.cnf proof.drat   // solve, output DRAT proof
./drat-trim input.cnf proof.drat -L proof.lrat   // transform DRAT proof to LRAT - fast
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Compressed DRAT and LRAT proofs

- Binary file instead of text file
  - Numbers stored as integers instead of strings
  - Implicit separators
- Variable byte length encoding for each literal, clause ID
  - A byte's first seven bits denote its actual value
  - A byte's last bit indicates if the number continues at the next byte
  - Makes proof independent of underlying integer domain (32 vs. 64 bit), saves space for small values

Adding Proof Support to Solvers

I wrote my own CDCL SAT solver. How can I let it produce UNSAT proofs?

DRAT:

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DRAT: super simple!

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Other formats:

- FRAT: Compromise between DRAT and FRAT at the developer’s discretion [2]
- DPR, LPR: Propagation Redundancy (PR) reasoning [3]
- VeriPB: Pseudo-Boolean reasoning [3]

Note: formally verified checkers are available for all these formats (sometimes translation-based)

Proof Production: The Parallel Case

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- Clause-sharing solvers:

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Before 2023: Large gap of trustworthiness between best sequential and best parallel (clause-sharing) solvers

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2023: LRAT-based proofs from clause-sharing solvers [12]

- Globally unique clause IDs without communication
  – for o original clauses and p solver threads, the i-th thread assigns clause IDs $o + i + kp$ ($k \in \mathbb{N}_0$)
- After solving, rewind the procedure, using "hints" of LRAT to trace dependencies of empty clause
- Funnel all clauses marked as required into a single, ordered proof file

Distributed Proof Production (1/2) [12]

Random 3-SAT formula, 180 variables. 4 notebook cores × 1.7 s. 300k dependencies (w/o orig. clauses).

Solving: Each thread derives and shares clauses, logs to partial proof file

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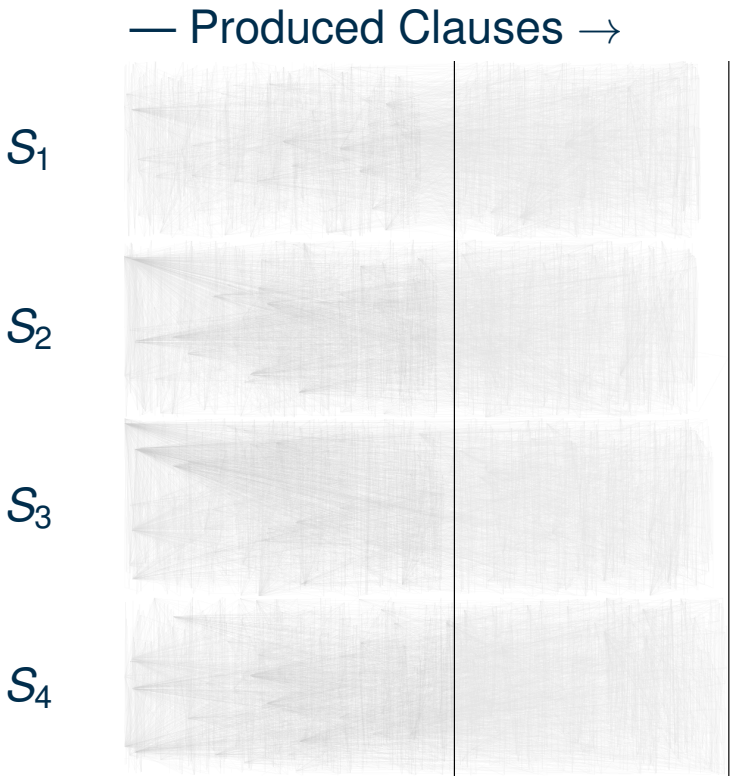

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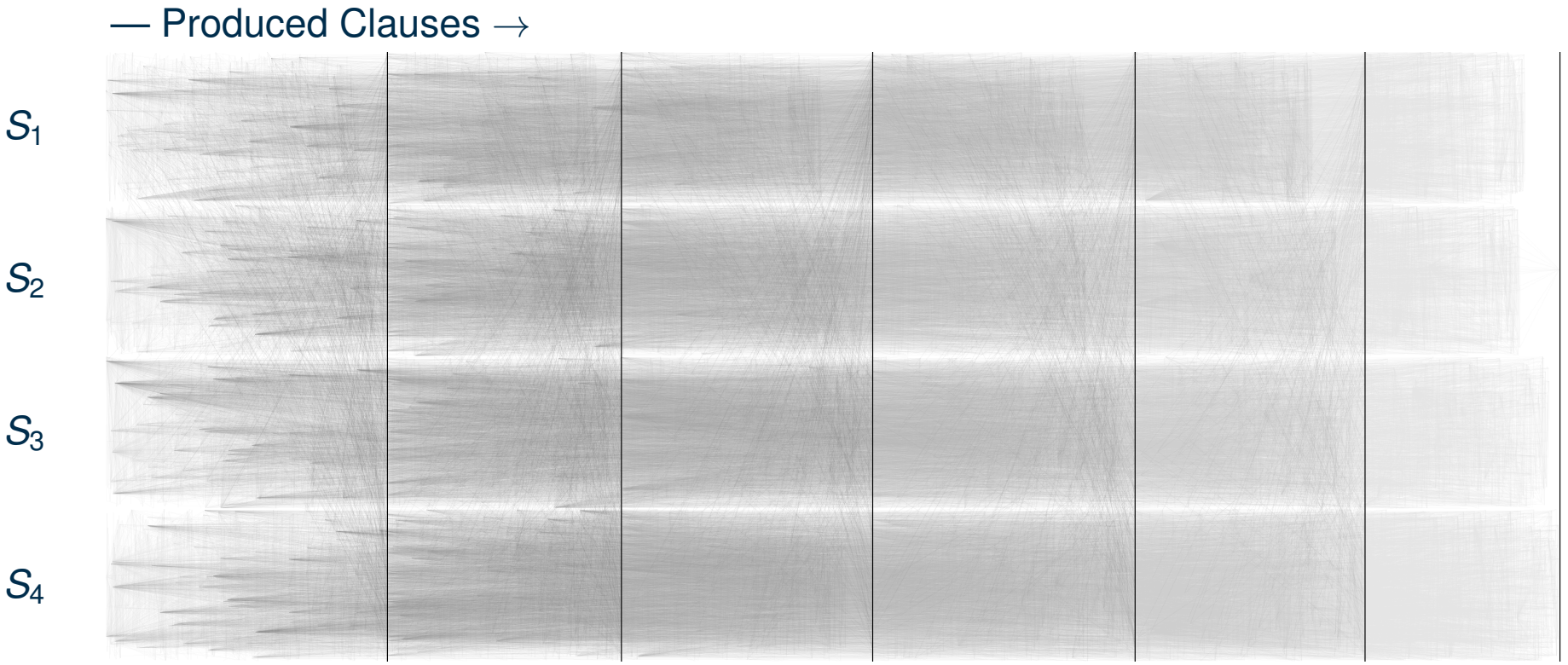

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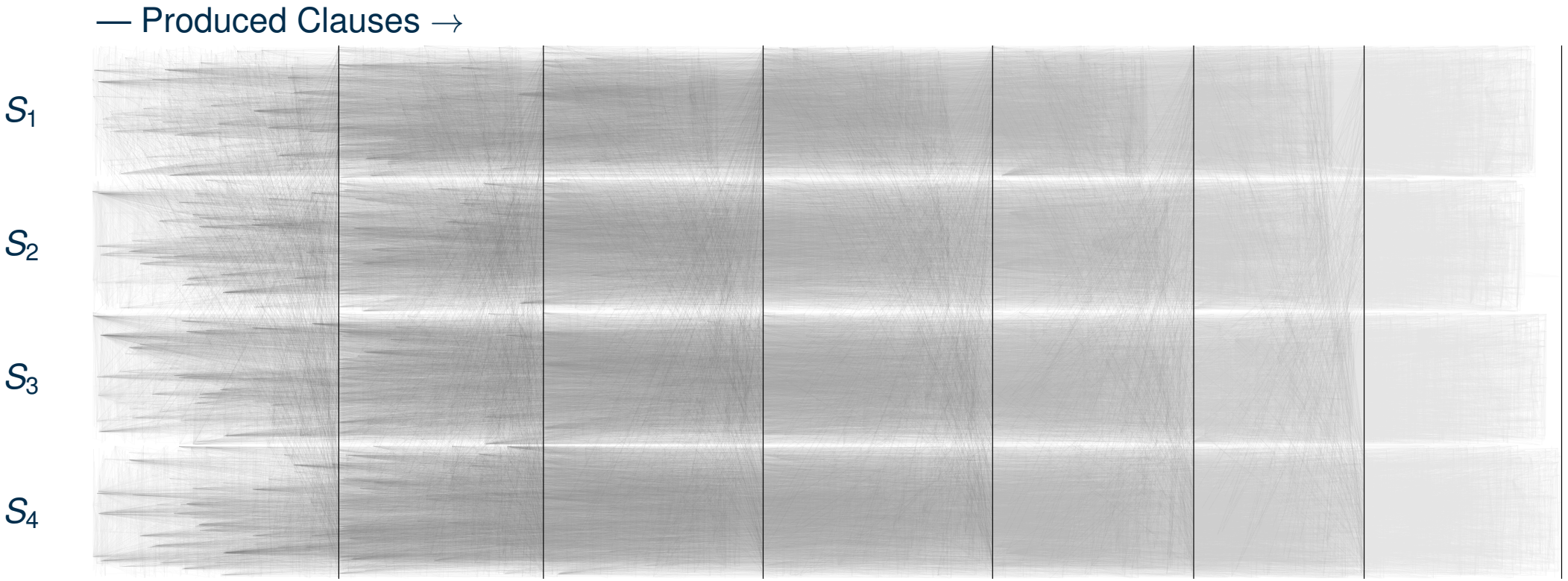

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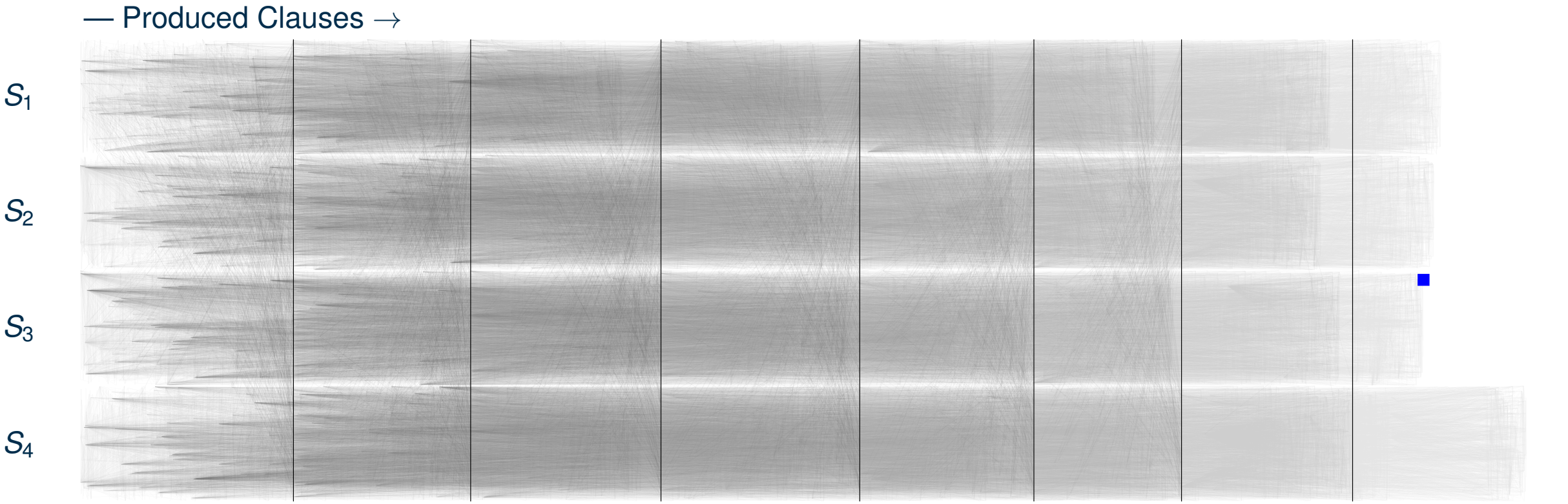

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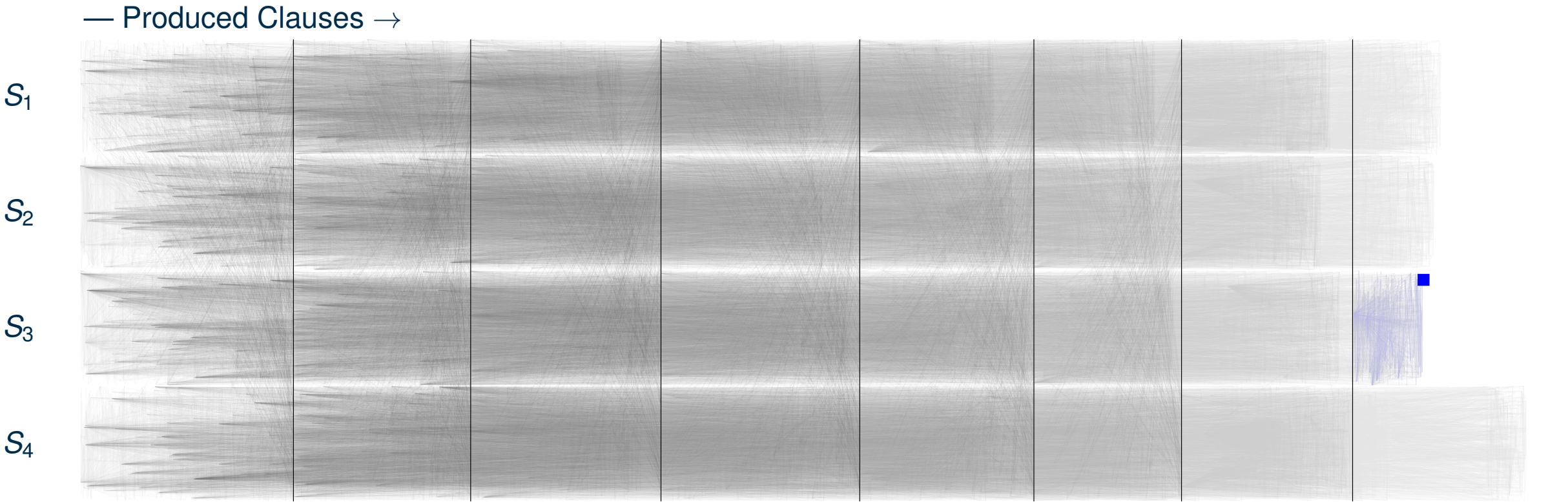

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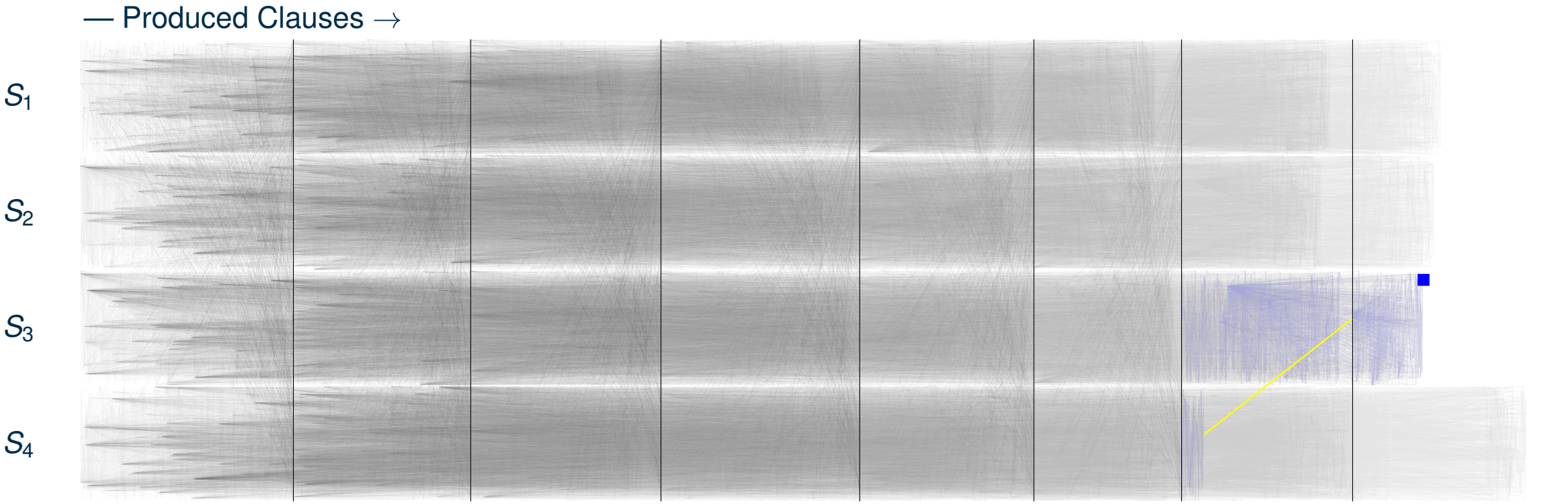

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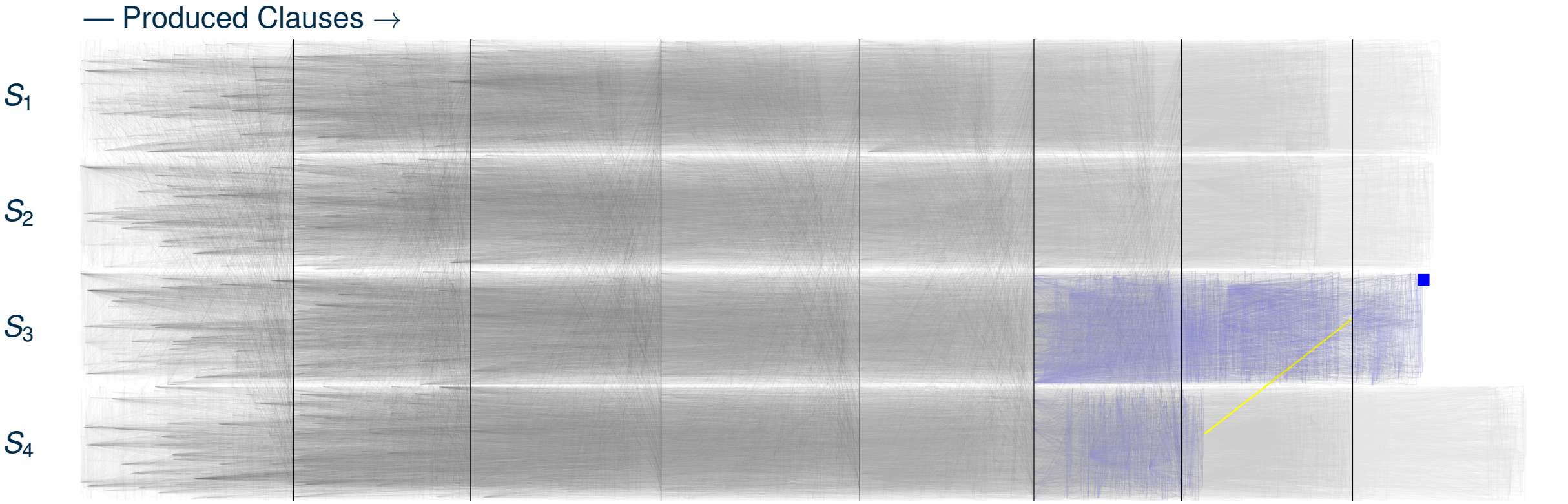

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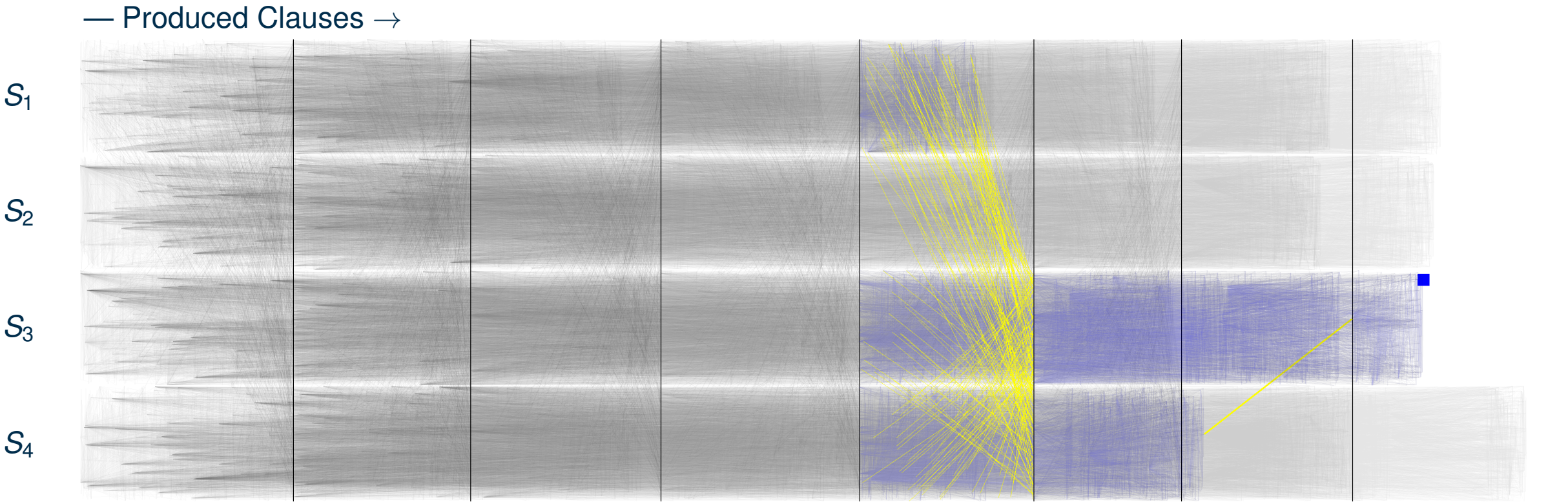

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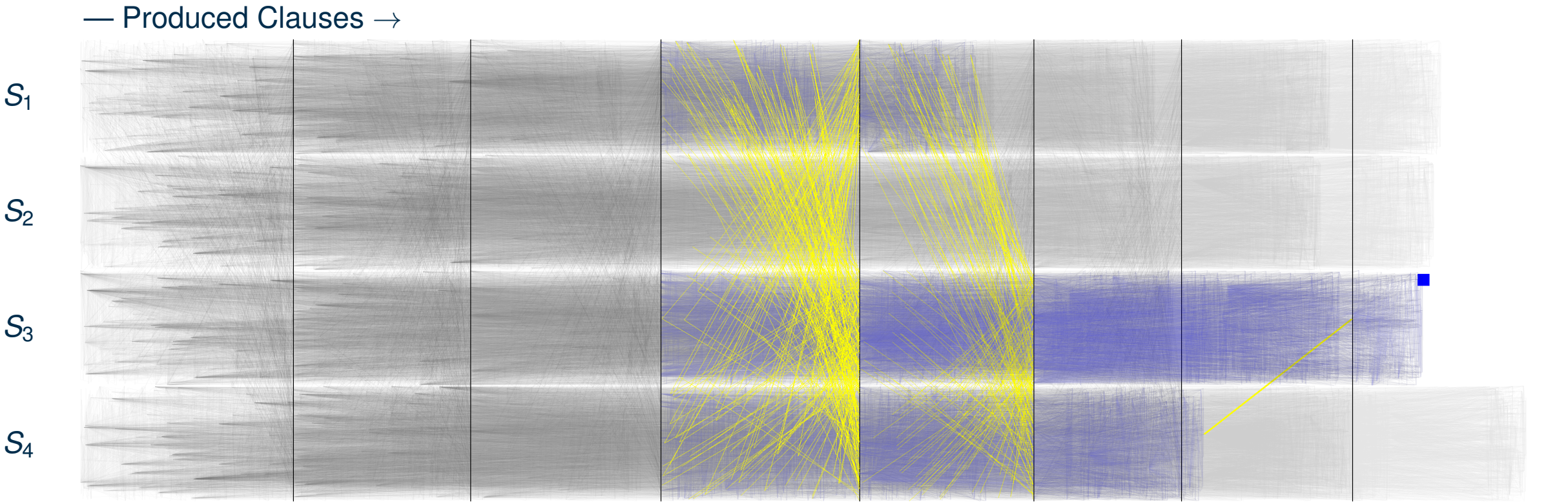

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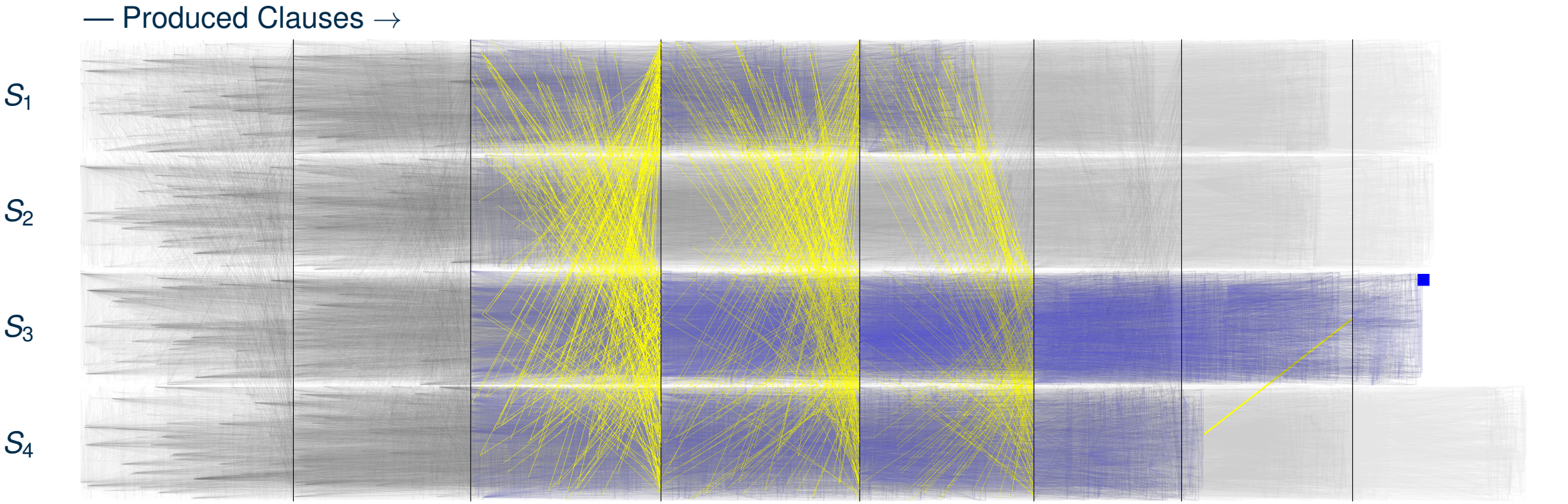

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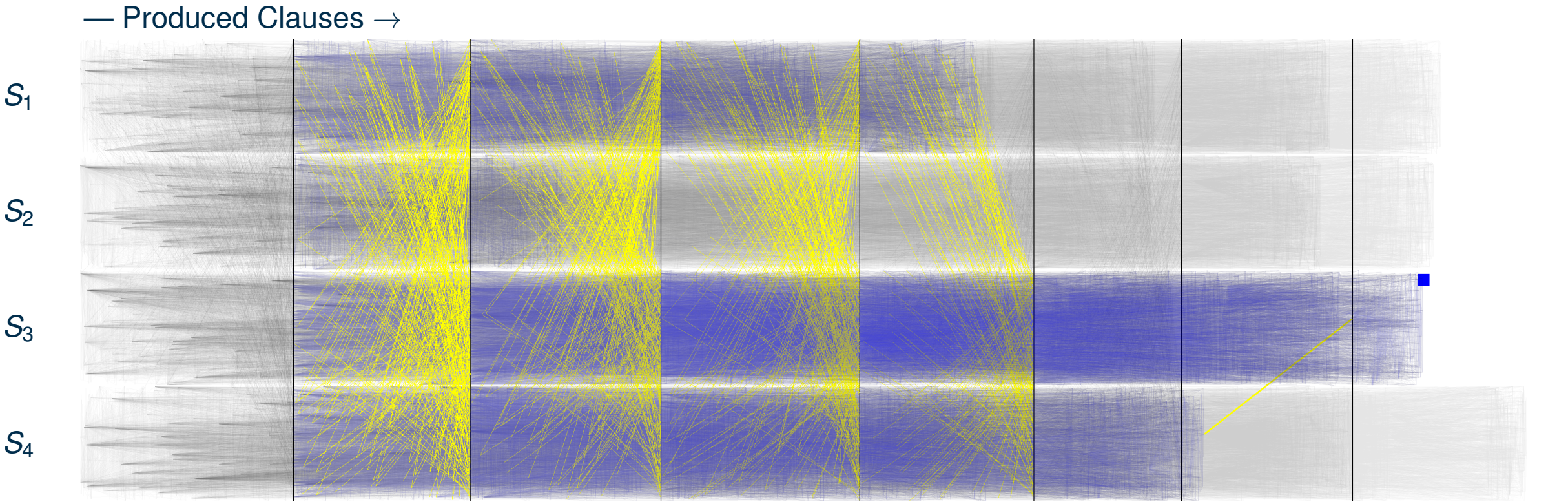

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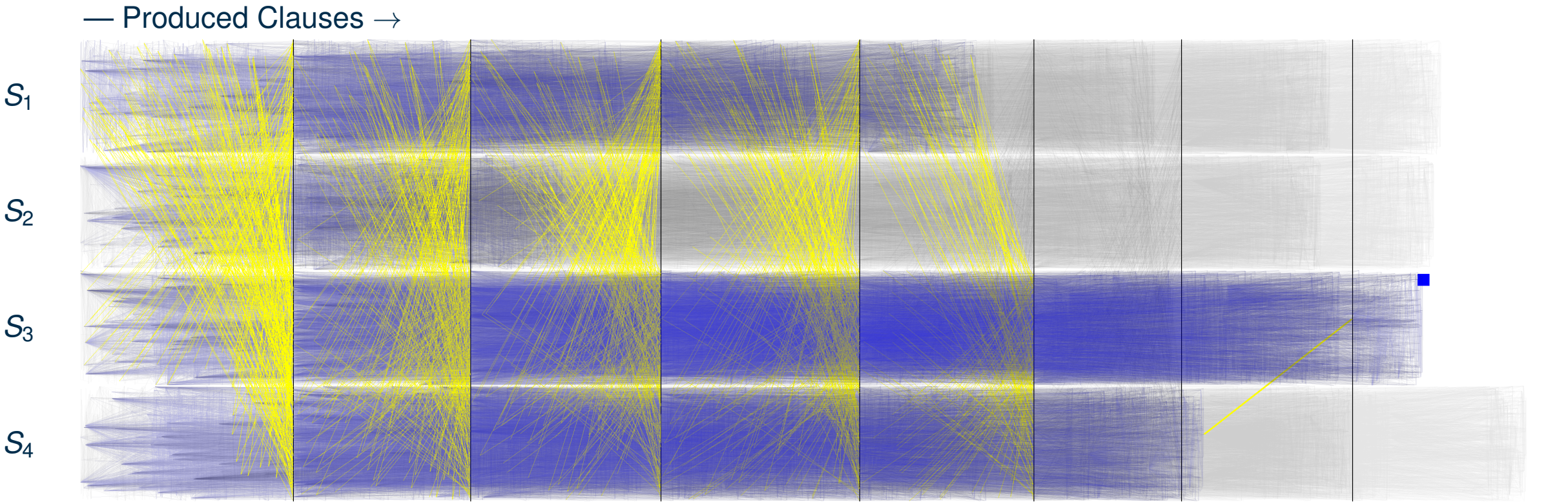

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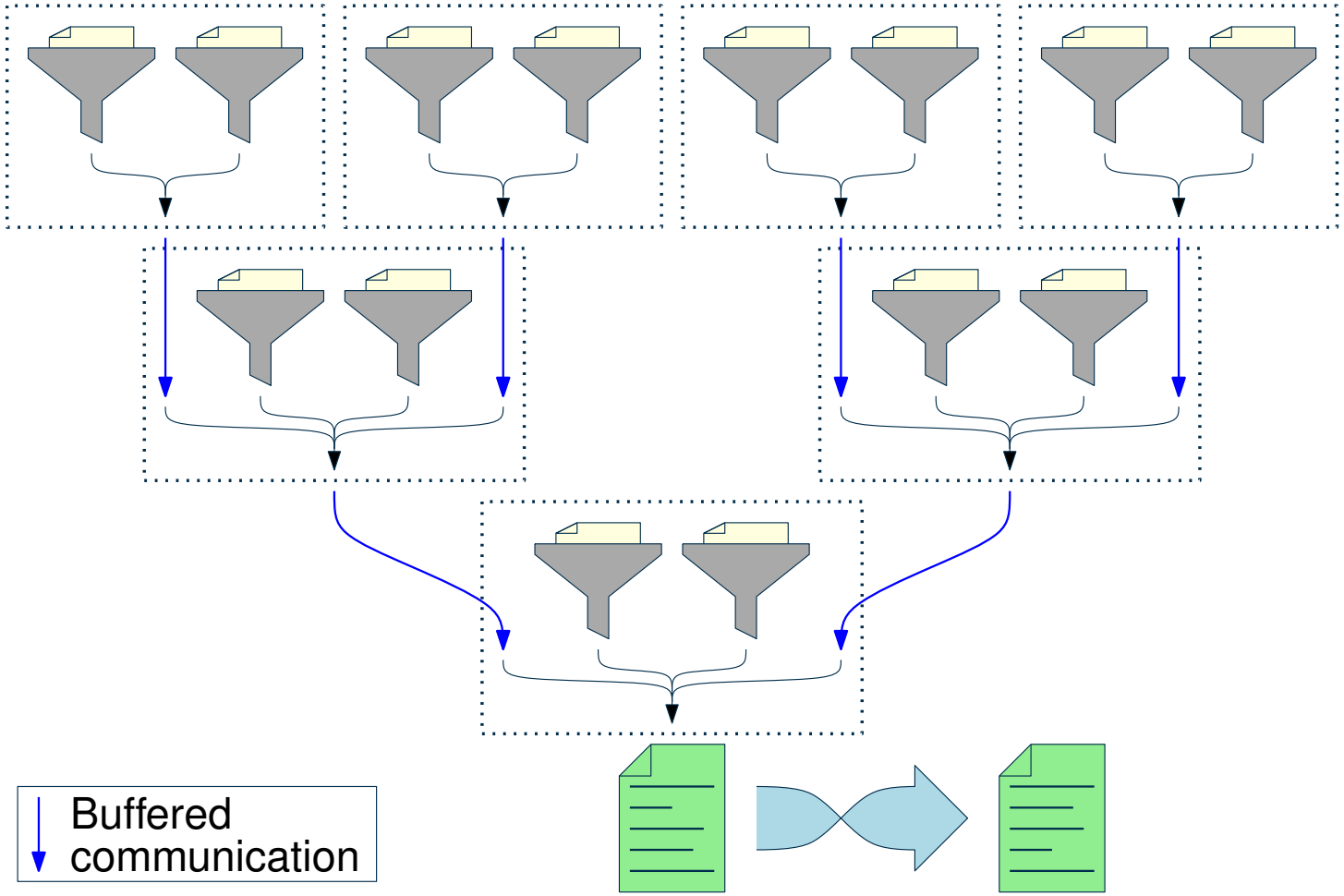
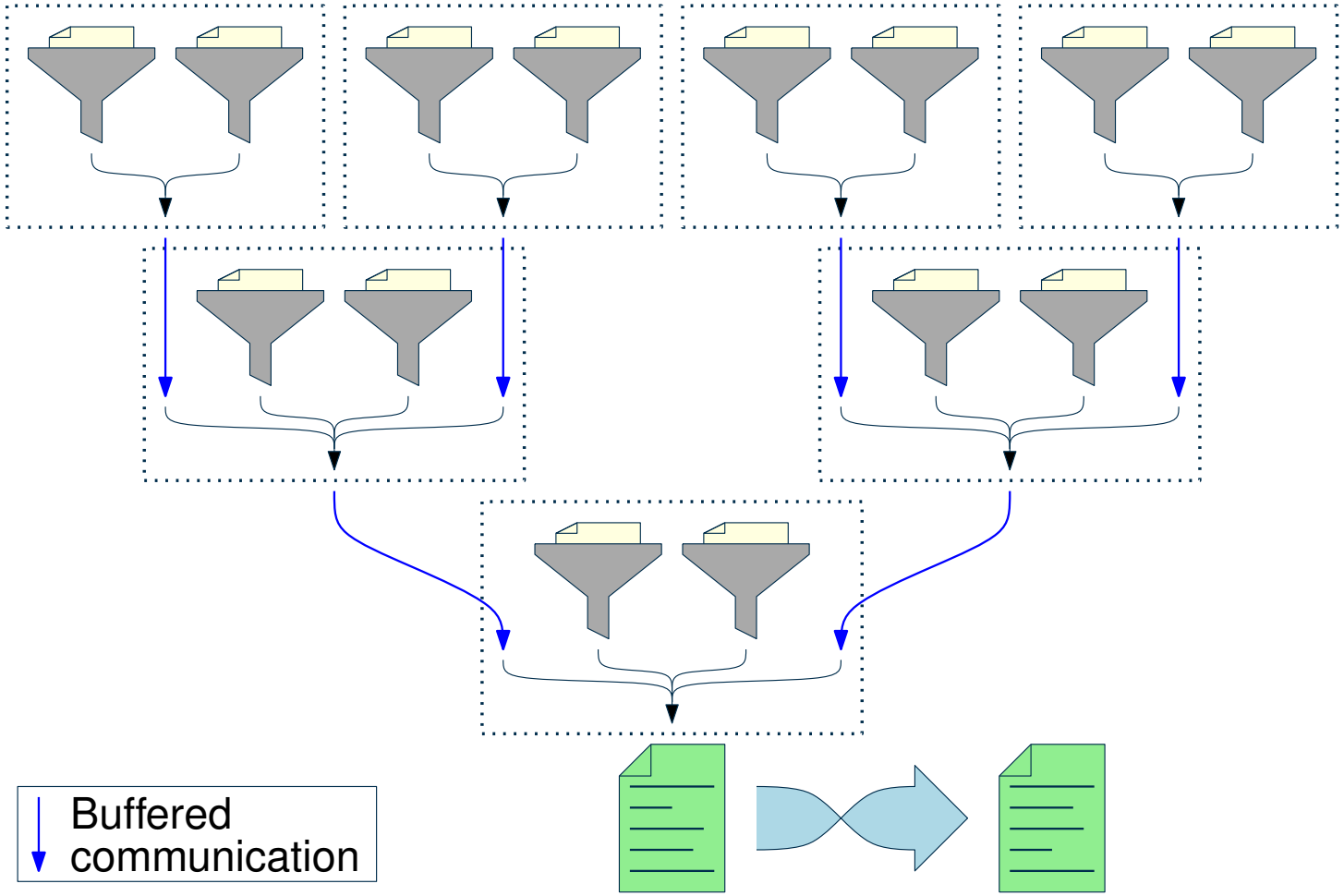
- Funnel required clause additions, still in reverse order, into singular proof file
- Hierarchical merging along tree



Distributed Proof Production (2/2) [12]

Merging:

- Funnel required clause additions, still in reverse order, into singular proof file
- Hierarchical merging along tree
- "Root process" writes output to file
  - Seeing an ID d_{ij} for the first time?
    $\Rightarrow$ write deletion of d_{ij} before writing the current statement!
  - Finally: Invert lines of proof file



Distributed Proof Production: Discussion

Results [11]: (using a hand-tailored LRUP checker operating on the inverted proof)

- Mean speedup of proof-emitting solver @ 1520 cores over sequential solver (solving times only): 17.5
  - Mean speedup of best MallobSat @ 1520 cores over sequential solver: 26.9
- On average, assembling and checking a proof takes $\approx 3\times$ solving time
  - Mean overhead of DRAT proof checking over sequential solving: $\approx 1\times$
- Pruning irrelevant clause additions reduces proof size by ≈ 30–$40\times$
- LRAT proof size: median 3.1 GB, mean 11.6 GB, maximum 233.9 GB

Bottleneck:

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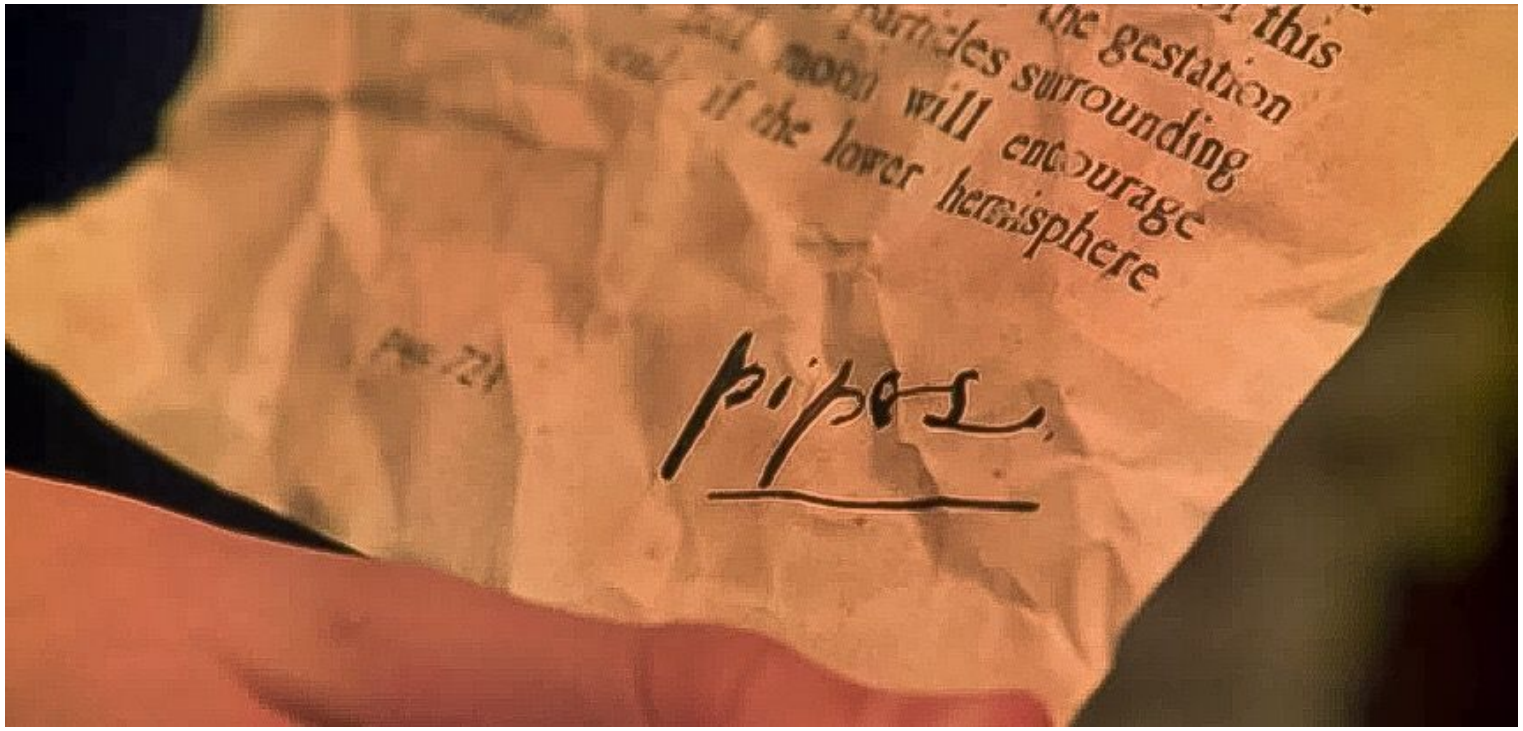
Bottleneck: Assembly and validation of a monolithic proof

- Proof creation throttled by I/O bandwidth at final process
- Checking can take very long
- The assembled proof's "corridor" of active clauses may no longer fit into RAM

Can we do better?

Hermione's Answer to More Scalable Trusted Solving



https://i.pinimg.com/originals/1b/3d/b6/1b3db639721eeafb188a3cc3060ff58b.jpg

Beyond Monolithic Proof Files

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mkfifo lratproof.pipe  // create “pipe” file
// Solve & check concurrently via pipe
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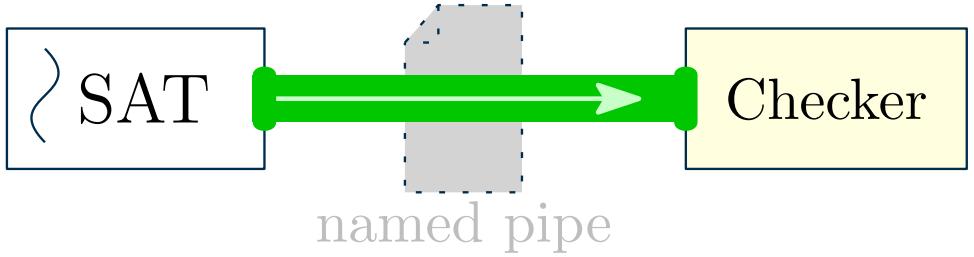

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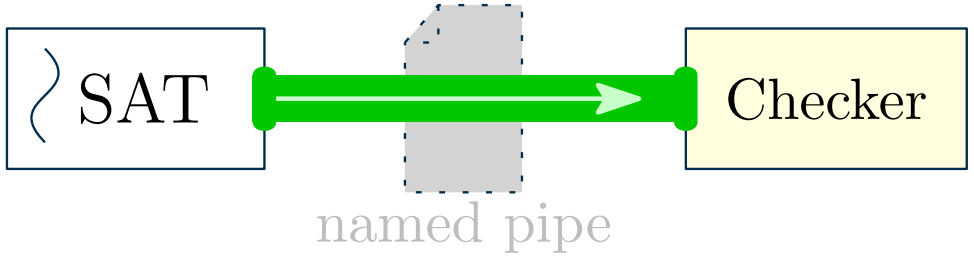

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- Solver streams proof output into a pipe (UNIX special file)
- Checker reads proof from pipe and checks it on-the-fly
  – checking is done as soon as solving is done!
- Almost no slowdown when running solver and checker on two hardware threads of the same core
- No disk I/O required, same program code as with normal files (execute `mkfifo` beforehand)
- Does not yield a persistent artifact to validate by independent parties

Clause Sharing with Real-time Checking [14]

- Run one checker process for each solver thread, mirroring its reasoning
- What about incoming shared clauses from another solver thread?

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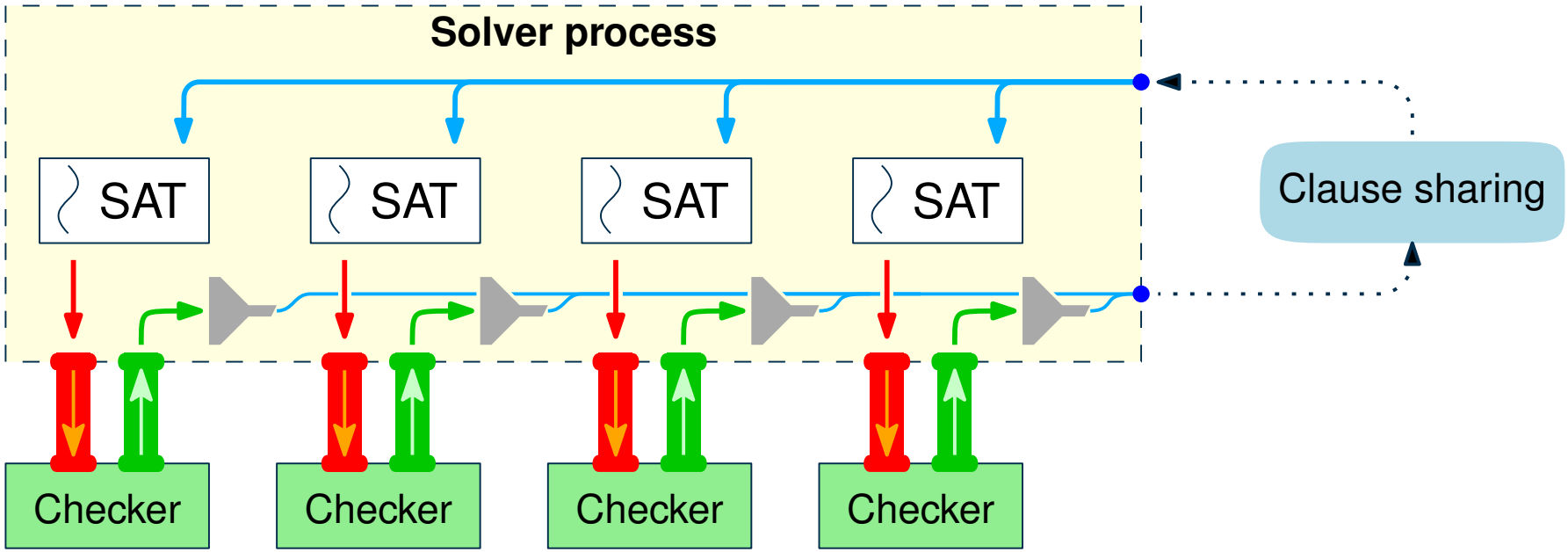
- Checkers sign each successfully checked clause c with cryptographic checksum based on shared secret K:
  $\mathcal{S}_K(c) = H_K(\texttt{LRUP\_ID}(c) \mid\mid c \mid\mid \mathcal{S}_K(F))$
  - H_K: Message Authentication Code (MAC), specifically 128-bit **SipHash** [1]
- Clauses are shared together with their checksums
- Each incoming clause c is forwarded to checker together with $\mathcal{S}_K(c)$
  $\Rightarrow$ Checker can validate that another trusted instance checked this clause!

Real-time Checking: Full Setup



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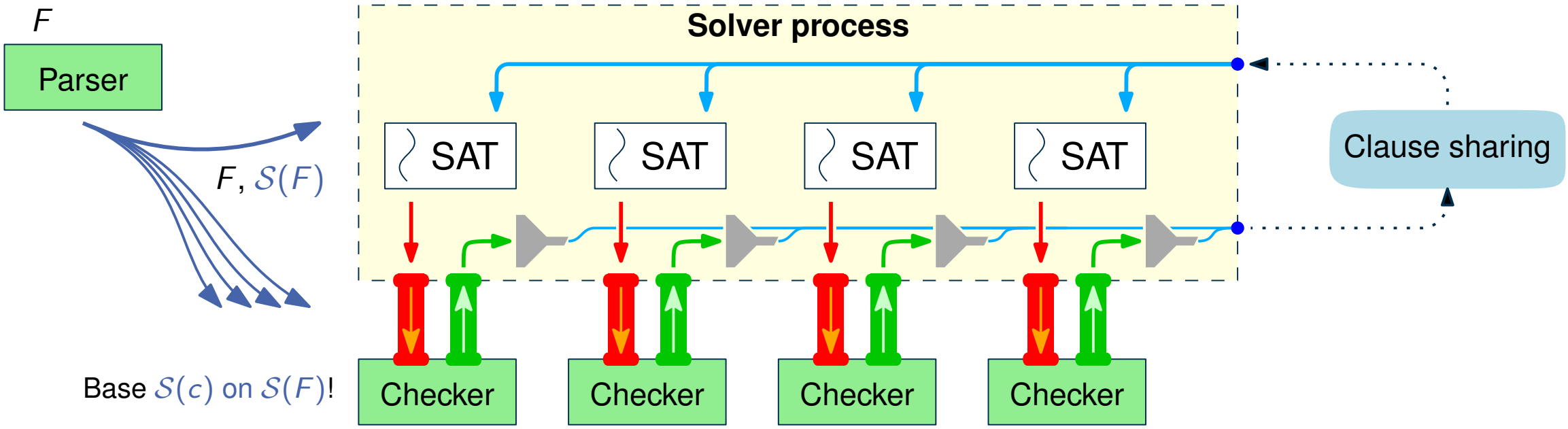

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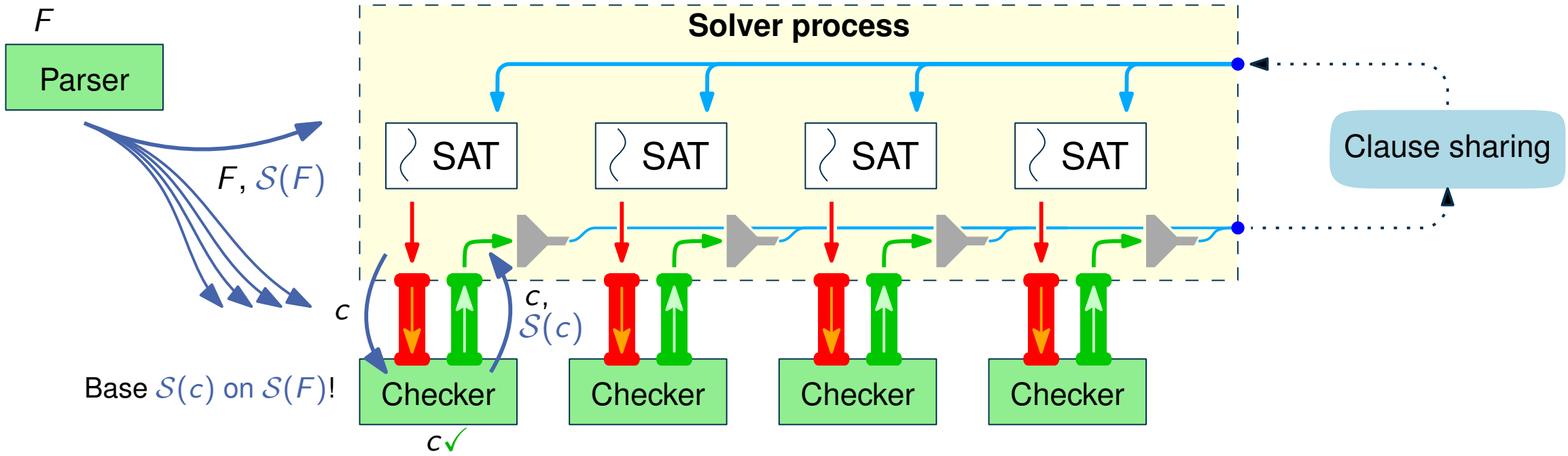

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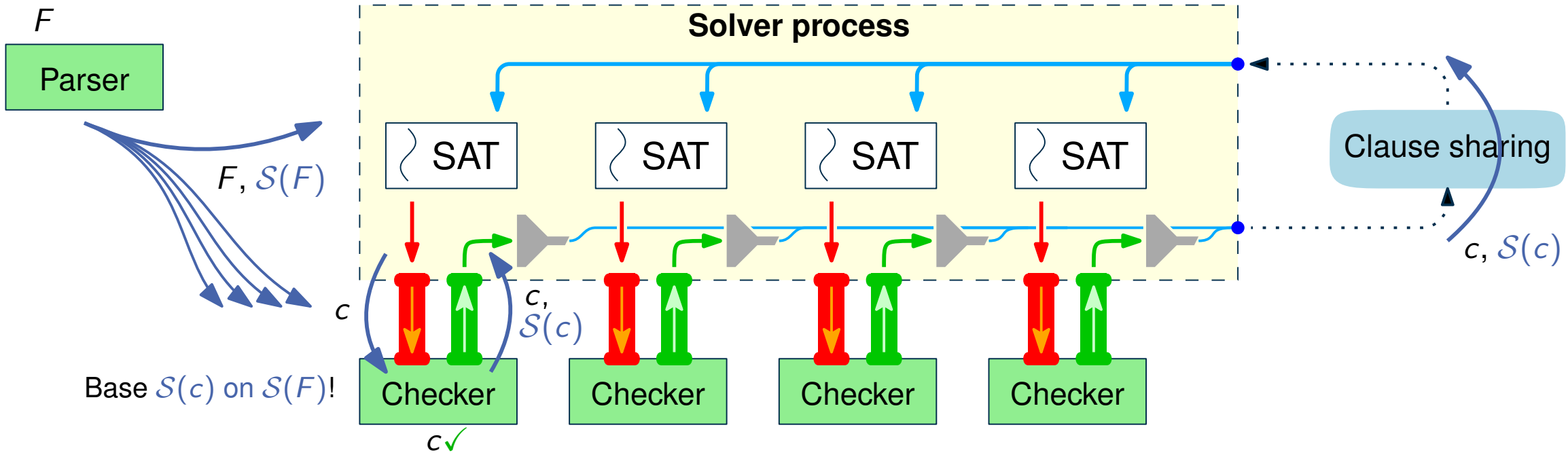

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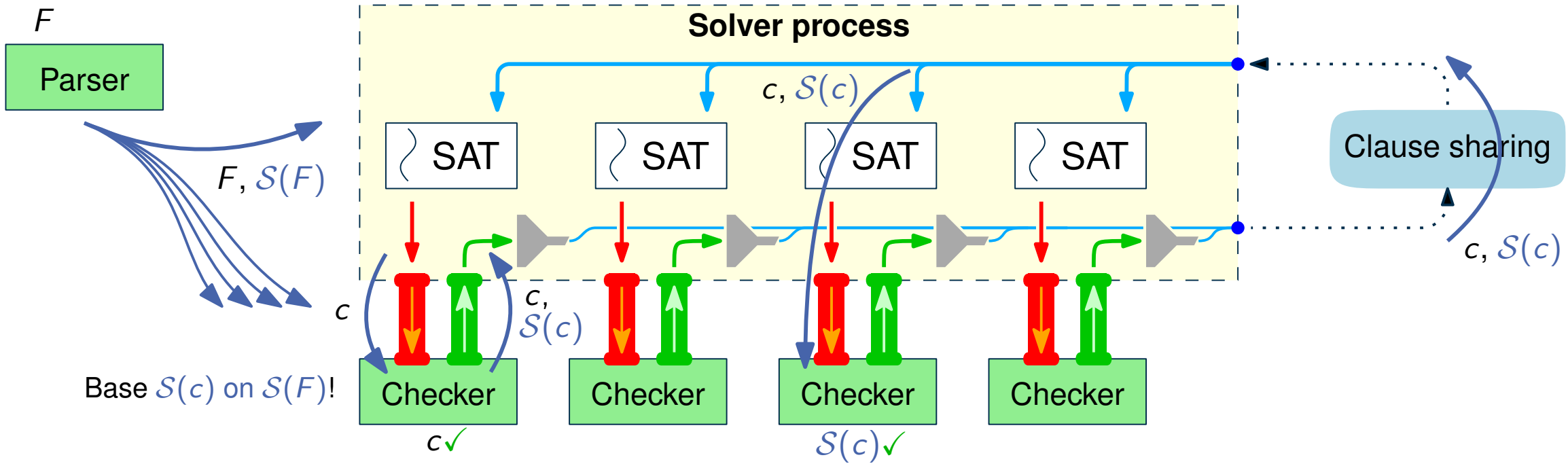

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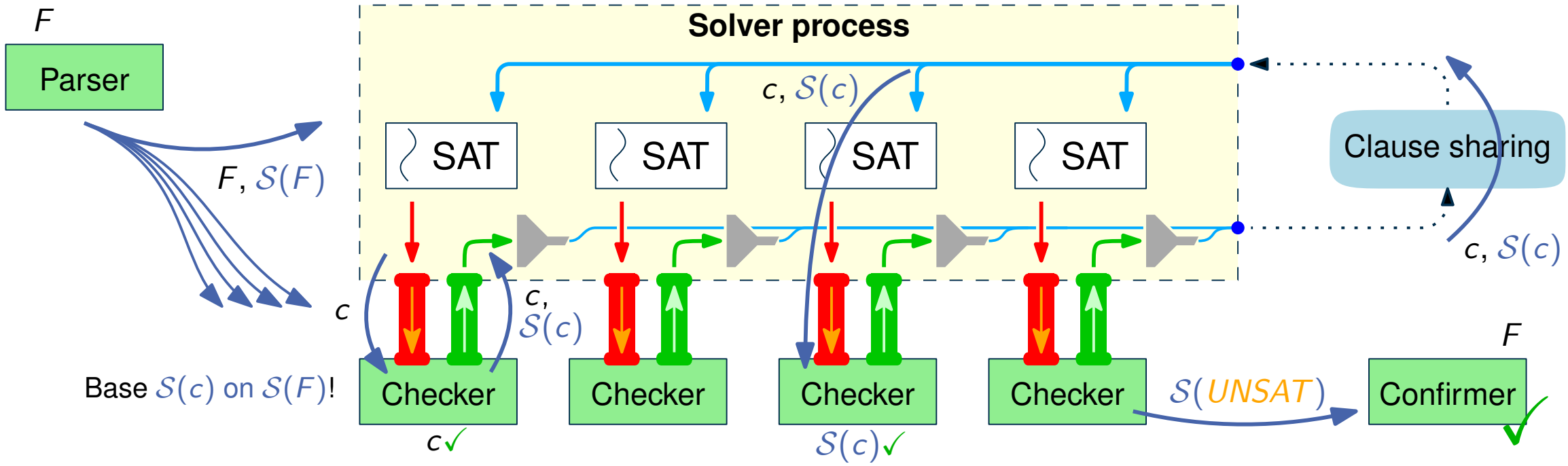

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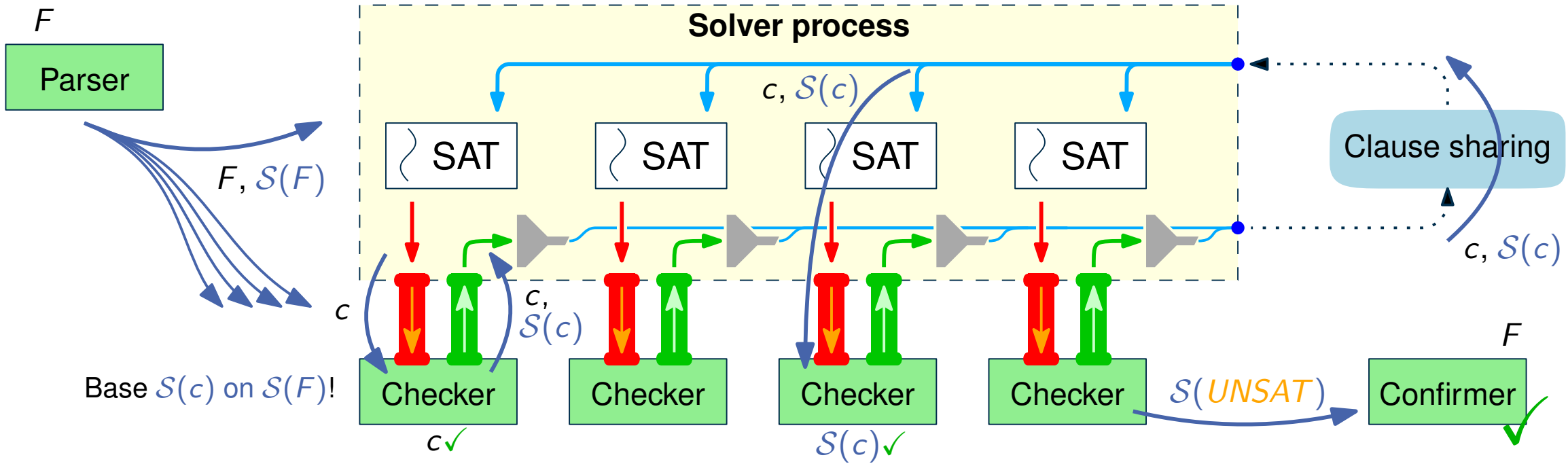
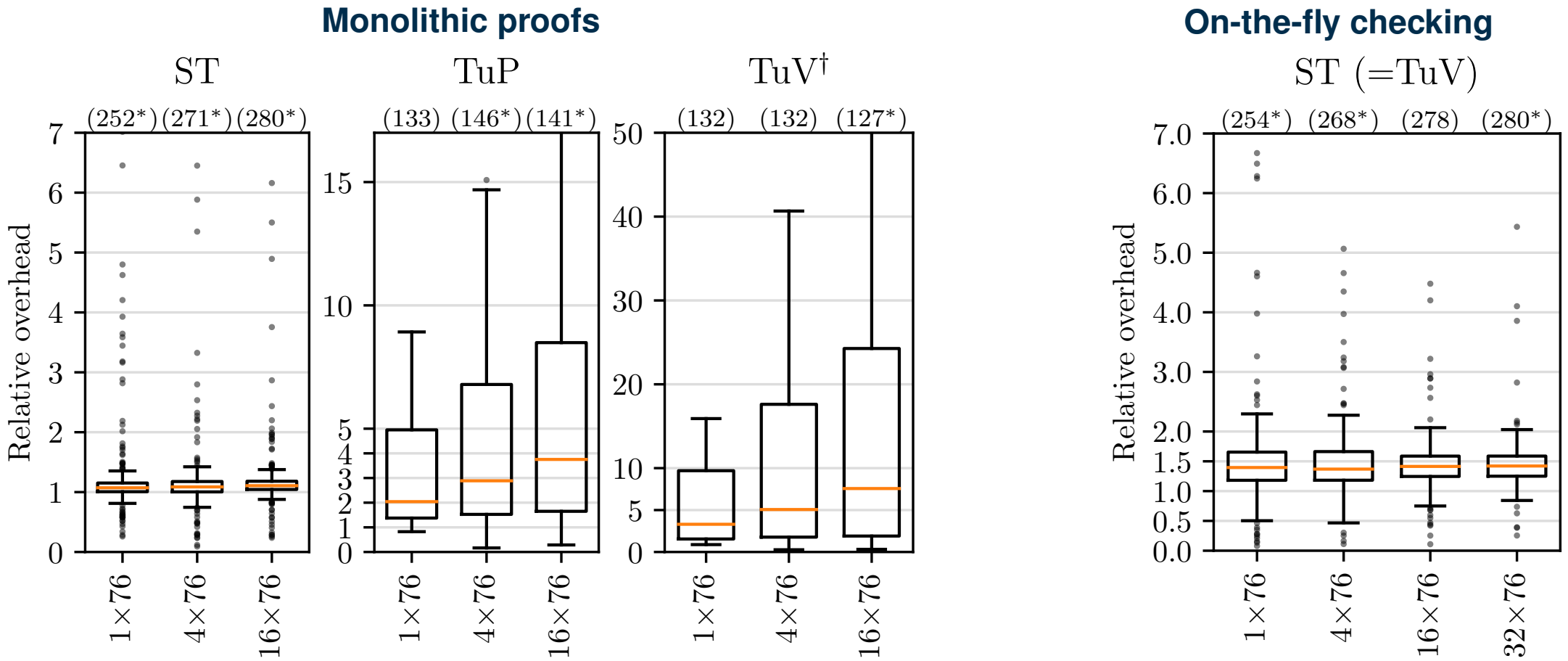


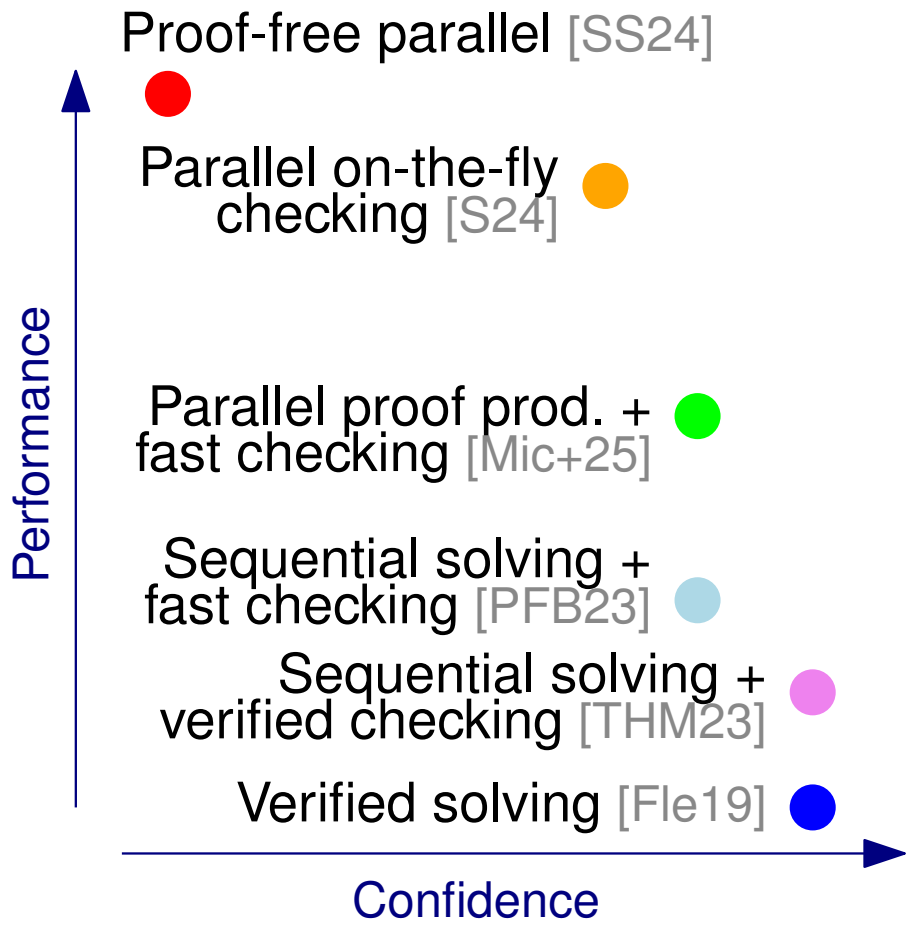
⇒ **We only need to trust our checksums and our parser, checkers, and confirmer.**
Not the SAT solvers, not the sharing logic, not the communication, . . .

Parallel Proof Tech: Scalability [14]



Overhead relative to proof-free solving time · ST: Solving time · TuP: Time until Proof present · TuV: Time until Validation done
76-core nodes · †Data extrapolated · *some data outside of displayed domain

Frontiers of Parallel Proof Technology



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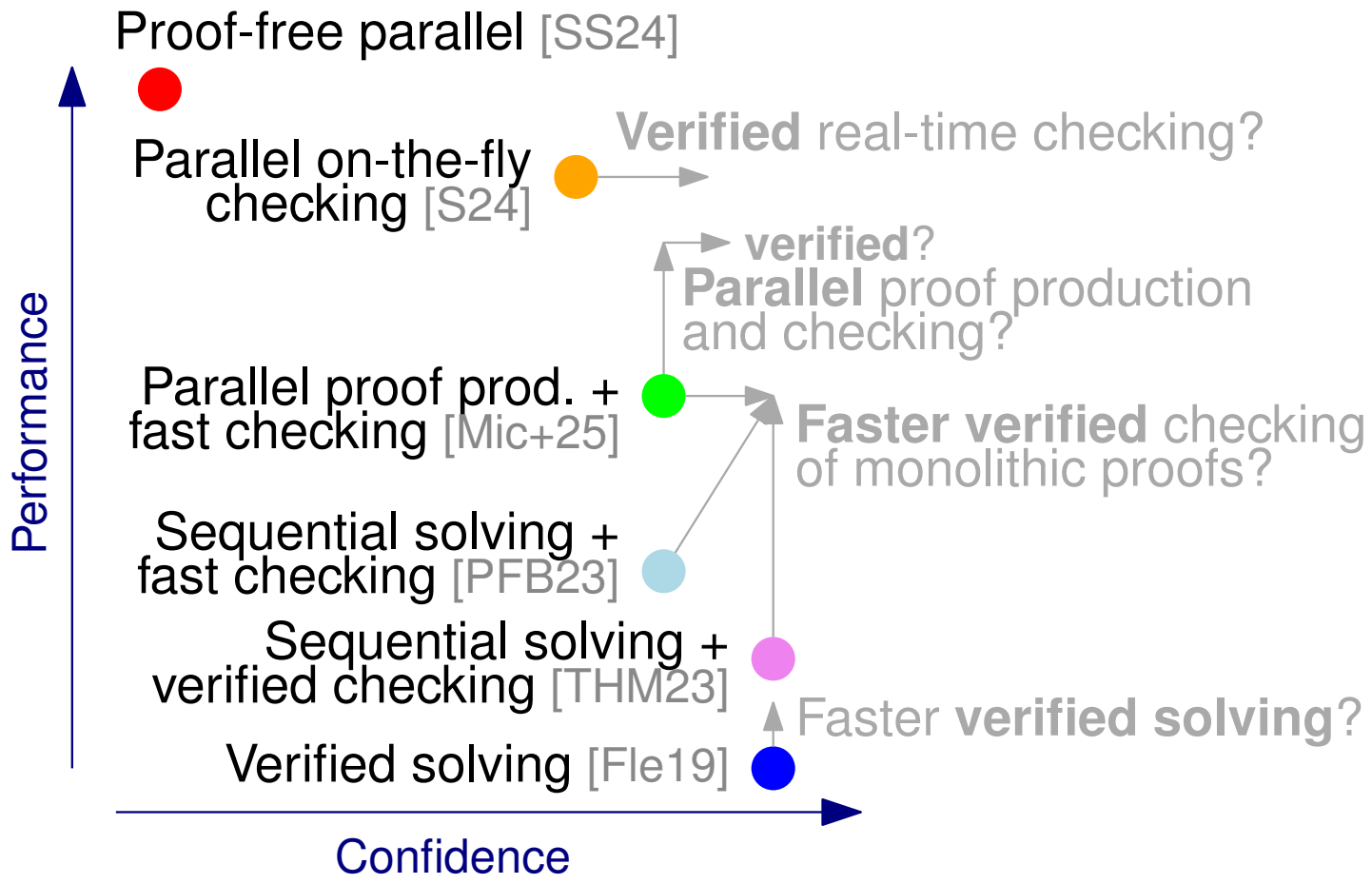

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- **Proof formats:** Trade-off between expressivity, checking efficiency, and solver development effort

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- Best of both worlds possible? Full scalability and persistent artifact?
- **Right now:** Rise of new proof formats (PR, PB, . . .) promising shorter proofs for some problems [3]
  – DRAT / LRAT is technically just as powerful but relies on variable addition for most powerful proofs
  - huge decision space, difficult to find a short proof
  - But: recent success in effective structured variable addition [6]

Recap

Proof Pragmatics & Parallel Proof Technology

- Propositional proof formats in practice
  - DRUP, DRAT, LRUP, LRAT
  - Time and memory complexity
  - Practical implementations
- Proof technology for parallel and distributed SAT solvers
  - Constructing monolithic proof files
  - Checking reasoning in real-time

Next Up: Applications of SAT solving

- Automated planning
- Bounded Model Checking

References I

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