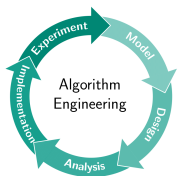


Practical SAT Solving

Lecture 3 – Elementary SAT Solving Algorithms

Ashlin Iser, [Dominik Schreiber](#) | May 4, 2025



Overview

Recap. Lecture 2

- Tractable Subclasses
- Constraint Encodings
- Encoding Techniques

Today's Topics: Elementary SAT Algorithms

- Local Search
- Resolution
- DP Algorithm
- DPLL Algorithm

Stochastic Local Search (SLS)

Minimize the Number of Unsatisfied Clauses

Start with a **random** complete variable assignment α :

0	0	1	0	1	1	0	0	1	1	0	1	0	1	0
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

Repeatedly **flip** variables in α to decrease the number of unsatisfied clauses:

0	0	1	0	1	0	0	0	1	1	0	1	0	1	0
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

Stochastic Local Search (SLS)

Properties of SLS Algorithms

Local search algorithms are **incomplete**: They **cannot show unsatisfiability!**

Challenges:

- Which variable should be flipped next?

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- Which variable should be flipped next?
 - select variable from an **unsatisfied clause**
 - select variable that **maximizes** the number of satisfied clauses
- How to avoid getting stuck in **local minima**?

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Challenges:

- Which variable should be flipped next?
 - select variable from an **unsatisfied clause**
 - select variable that **maximizes** the number of satisfied clauses
- How to avoid getting stuck in **local minima**?
 - **Randomization!**

Classic SLS Algorithms

GSAT

Greedy local search algorithm

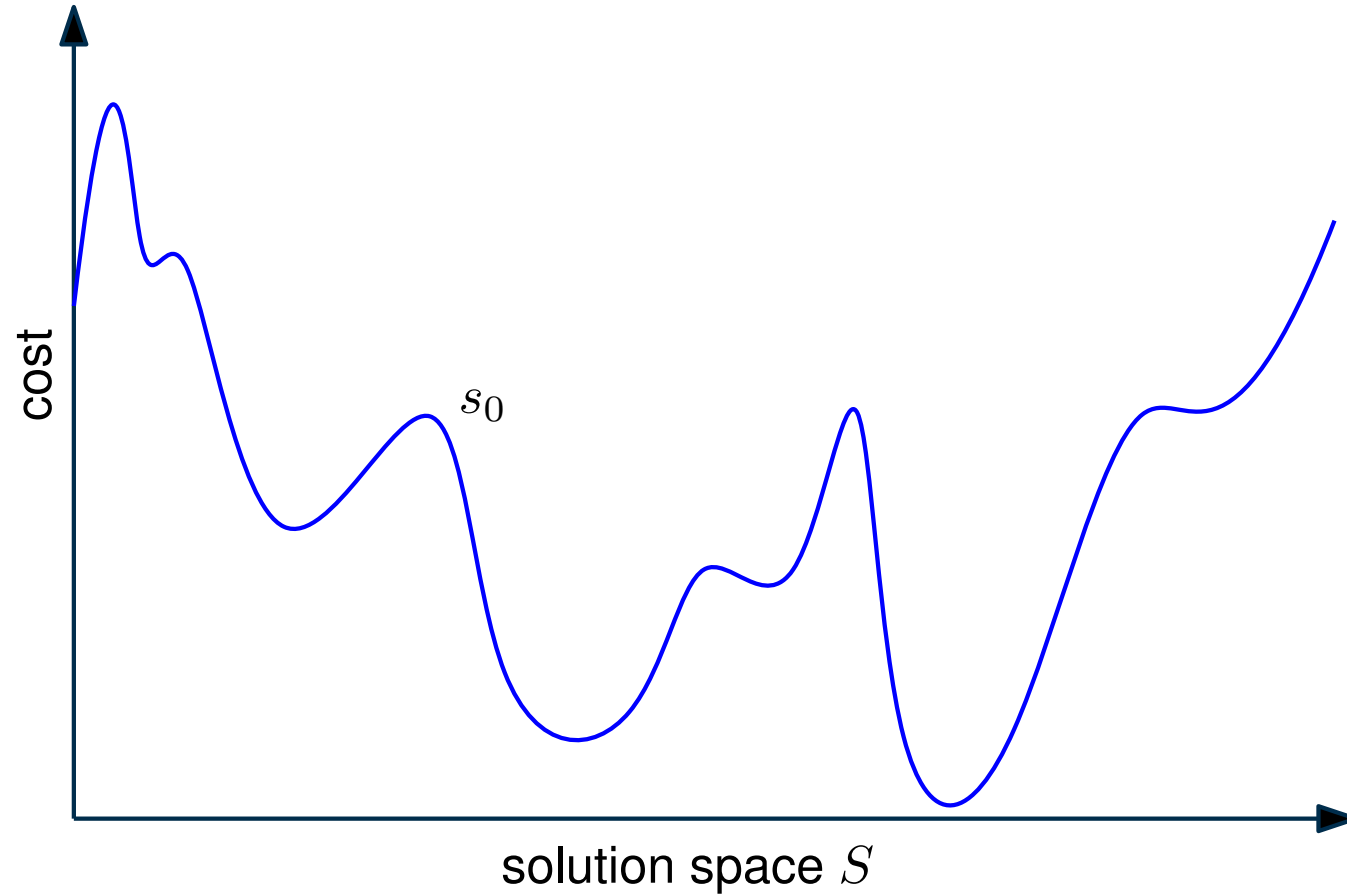
Algorithm: GSAT

Input: ClauseSet S

Output: Assignment α , or Nothing

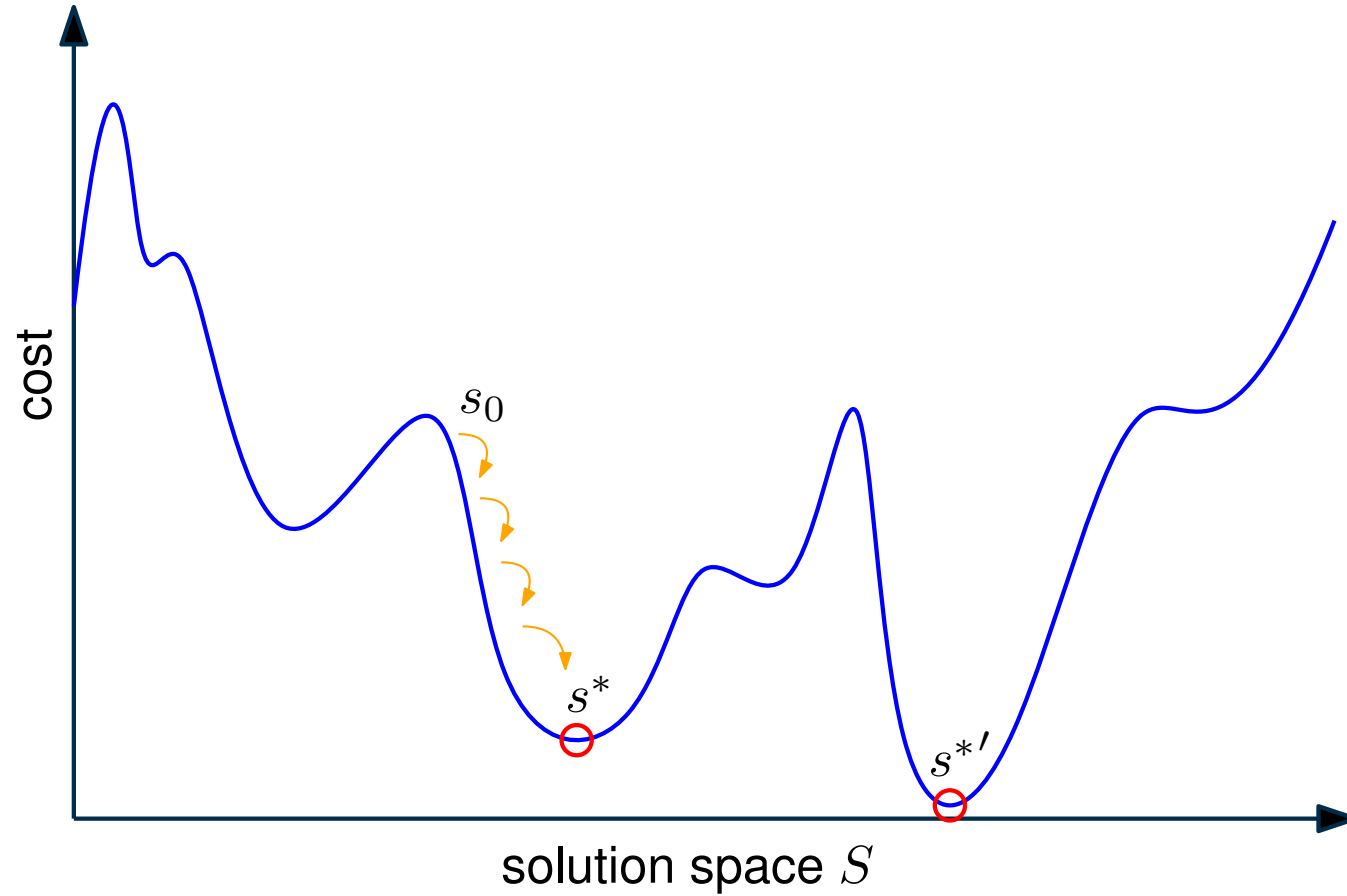
```
1 for  $i = 1$  to  $MAX\_TRIES$  do
2    $\alpha$  = random-assignment to variables in  $S$ 
3   for  $j = 1$  to  $MAX\_FLIPS$  do
4     if  $\alpha$  satisfies all clauses in  $S$  then return  $\alpha$ 
5      $x$  = variable that produces least number of unsatisfied clauses
        when flipped
6     flip  $x$ 
7 return Nothing // no solution found
```

SLS: Local Minima



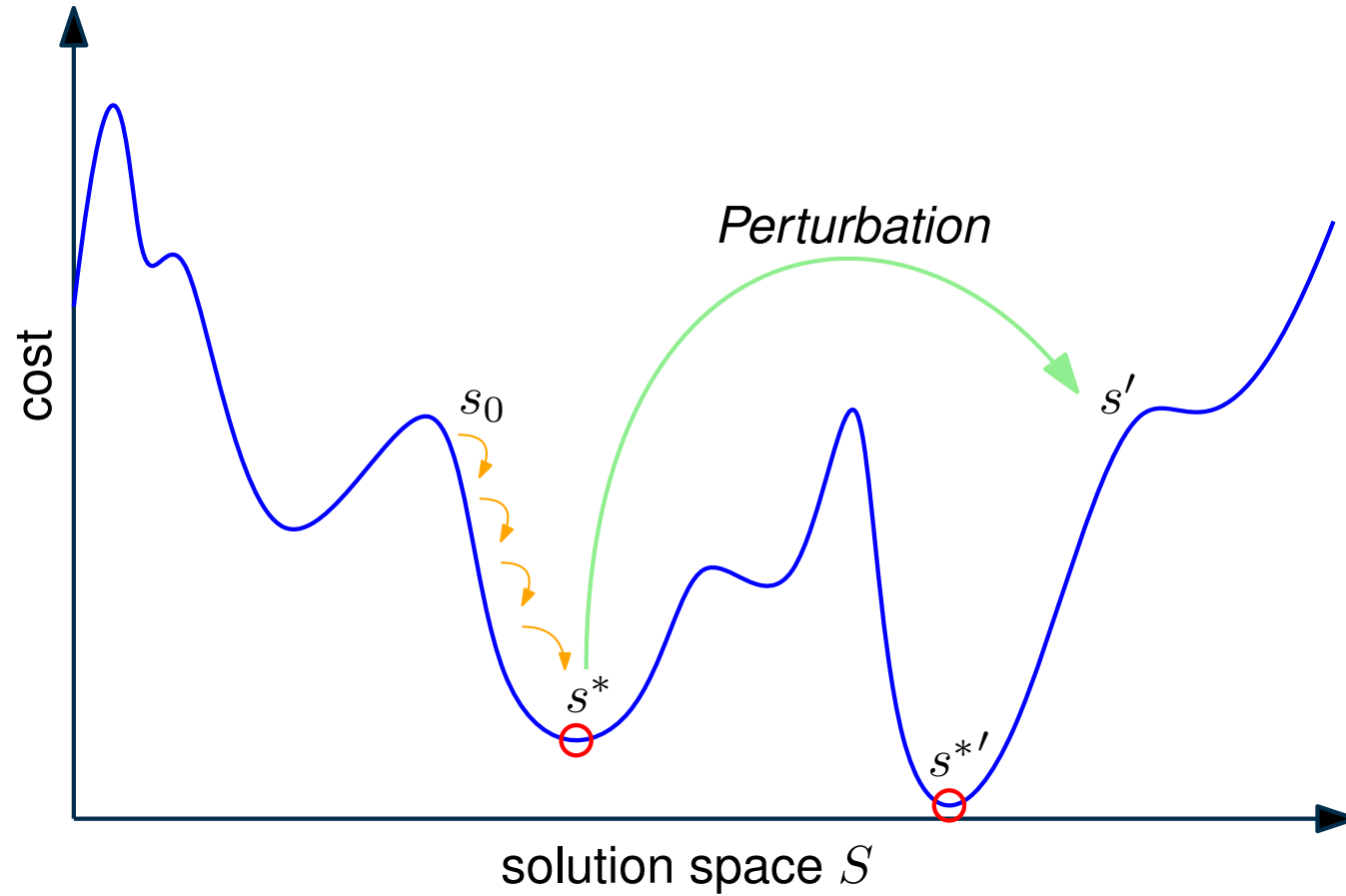
[Illustration Adapted from: Alan Mackworth, UBC, Canada]

SLS: Local Minima



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SLS: Local Minima



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Classic SLS Algorithms

WalkSAT

Variant of GSAT

Try to avoid local minima by introducing random noise.

Algorithm: WalkSAT(S)

```
1 for  $i = 1$  to  $MAX\_TRIES$  do
2    $\alpha$  = random-assignment to variables in  $S$ 
3   for  $j = 1$  to  $MAX\_FLIPS$  do
4     if  $\alpha$  satisfies all clauses in  $S$  then return  $\alpha$ 
5      $C$  = random unsatisfied clause in  $S$ 
6     if by flipping an  $x \in C$  no new unsatisfied clauses emerges
7       then flip  $x$ 
8     else with probability  $p$  flip an  $x \in C$  at random
9     otherwise, flip a variable that changes the least number of
10    clauses from satisfied to unsatisfied
11  return Nothing // no solution found
```

SLS: Important Notions

Consider a flip taking α to α'

breakcount number of clauses satisfied in α , but not satisfied in α'

makecount number of clauses not satisfied in α , but satisfied in α'

diffscore # unsatisfied clauses in α – # unsatisfied clauses in α'

Typically, breakcount, makecount, and/or diffscore are used to select the variable to flip.

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Recap using new nomenclature

GSAT select variable with highest diffscore

WalkSAT select variable with minimal breakcount

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Legacy of SLS

- Extremely successful and popular in early days of SAT
 - SLS outperformed early resolution-based solvers, e.g., based on DP or DPLL
 - for example, state of the art engine for automated planning in the 90s

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 - Faster, more reliable, and **complete!**

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 - for example, state of the art engine for **automated planning** in the 90s
- Today, sophisticated resolution-based systematic search solvers dominate in most practical applications
 - Faster, more reliable, and **complete!**
- Still useful as a component in more complex solvers
 - Part of (parallel) **algorithm portfolios**
 - Control **branching heuristics** in complete search algorithms
 - Detection of autarkies in **formula simplification** algorithms
 - In combination with complete solvers for optimization problems (e.g., **MaxSAT**)

Recap

Elementary Algorithms

- Local Search
 - Examples: GSAT, WalkSAT
 - Terminology: breakcount, makecount, diffscore

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Next Up

- Resolution

Resolution

The Resolution Rule

$$\frac{P_1 \cup \{x\} \quad P_2 \cup \{\neg x\}}{P_1 \cup P_2}$$

Resolution is a logical inference rule, which infers a **conclusion** (resolvent) from given **premises** (input clauses).

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Example (Resolution)

$$\{x_1, x_3, \neg x_7\}, \{\neg x_1, x_2\} \vdash \{x_3, \neg x_7, x_2\}$$

$$\{x_4, x_5\}, \{\neg x_5\} \vdash \{x_4\}$$

$$\{x_1, x_2\}, \{\neg x_1, \neg x_2\} \vdash \{x_1, \neg x_1\}$$

$$\{x_1\}, \{\neg x_1\} \vdash \{\}$$

(Fact)

(Tautological Resolvent)

(Empty Clause)

Resolution

Theorem: Resolution is sound

Given a CNF formula F with two resolvable clauses $C_1, C_2 \subseteq F$ with resolvent $R(C_1, C_2)$, the following holds:

$$F \equiv F \wedge R(C_1, C_2)$$

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Proof

Let $C_1 := \{x\} \cup P_1$ and $C_2 := \{\neg x\} \cup P_2$ such that $R(C_1, C_2) = P_1 \cup P_2 =: D$.

Soundness: $F \vdash F \wedge D \implies F \models F \wedge D$

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Any satisfying assignment ϕ of F satisfies both C_1 and C_2 . If ϕ satisfies x , then it satisfies some literal in P_2 . Otherwise, ϕ satisfies $\neg x$ and thus satisfies some literal in P_1 . As such, ϕ also satisfies D .

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Equivalence: $F \vdash F \wedge D \implies F \wedge D \models F$

Since D does not introduce new variables, $F \wedge D$ can not have more satisfying assignments than F .

Resolution

Resolution is sound and Refutation-complete

- If we manage to infer the **empty clause** from a CNF formula F , then F is unsatisfiable. (sound)
- If F is unsatisfiable, then there **exists a refutation by resolution**. (refutation-complete)
- **Not all possible** consequences of F can be derived by resolution. (“only” refutation-complete)

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Resolution Proof

A **resolution proof** for F is a sequence of clauses $\langle C_1, C_2, \dots, C_{k-1}, C_k = \emptyset \rangle$ where each C_i is either an original clause of F or a resolvent of two earlier clauses.

Example (Resolution Proof)

$$\begin{aligned} F &= \{x_1, x_2\}, \{\neg x_1, x_2\}, \{x_1, \neg x_2\}, \{\neg x_1, \neg x_2\} && \text{(Formula)} \\ &\equiv \{x_1, x_2\}, \{\neg x_1, x_2\}, \{x_1, \neg x_2\}, \{\neg x_1, \neg x_2\}, \{x_2\}, \{\neg x_2\}, \{\} && \text{(Refutation)} \end{aligned}$$

Saturation Algorithm

Algorithm: Saturation Algorithm

Input: CNF formula F

Output: {SAT, UNSAT}

```
1 while true do
2    $R := \text{resolveAll}(F)$ 
3   if  $R \cap F \neq R$  then  $F := F \cup R$ 
4   else break
5 if  $\perp \in F$  then return UNSAT
6 else return SAT
```

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Properties

- sound and complete – always terminates and answers correctly
- exponential time and space complexity

Unit Propagation

Unit Resolution

Resolution where at least one of the resolved clauses is a **unit clause**, i.e., has size one.

Example (Unit Resolution)

$$R((x_1 \vee x_7 \vee \neg x_2 \vee x_4), (x_2)) = (x_1 \vee x_7 \vee x_4)$$

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Unit Propagation

Apply unit resolution until fixpoint is reached.

Example (Unit Propagation)

Usually, we are only interested in the inferred facts (unit clauses) and conflicts (empty clauses).

$$\{x_1, x_2, x_3\}, \{x_1, \neg x_2\}, \{\neg x_1\} \vdash_1 \{\neg x_2\}, \{x_3\}$$

Recap

Elementary Algorithms

- Local Search
 - Examples: GSAT, WalkSAT
 - Terminology: breakcount, makecount, diffscore
- Resolution
 - Soundness and Completeness
 - Saturation Algorithm (Exponential Complexity)
 - Unit Propagation

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Next Up

Improving upon saturation-based resolution: Davis Putnam (DP) Algorithm

Davis-Putnam Algorithm (Davis & Putnam, 1960)

Presented in 1960 as a SAT procedure for first-order logic.

Deduction Rules of DP Algorithm

- **Unit Resolution:** If there is a unit clause $C = \{\bar{x}\} \in F$, simplify all other clauses containing x
- **Pure Literal Elimination:** If a literal x never occurs negated in F , add clause $\{x\}$ to F
- **Case Splitting:** Put F in the form $(A \vee x) \wedge (B \vee \neg x) \wedge R$, where A , B , and R are clause sets free of x . Replace F by the clausification of $(A \vee B) \wedge R$

Apply above deduction rules (prioritizing rules 1 and 2) until one of the following situations occurs:

- $F = \emptyset \rightarrow \text{SAT}$
- $\emptyset \in F \rightarrow \text{UNSAT}$

Davis-Putnam Algorithm

Example (DP Algorithm)

$$F = \{\{x, y, \neg z, u\}, \{\neg x, y, u\}, \{x, \neg y, \neg z\}, \{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}, \{\neg x, \neg y, u\}\}$$

(Split by x)

Davis-Putnam Algorithm

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$$A = \{\{y, \neg z, u\}, \{\neg y, \neg z\}\} \quad B = \{\{y, u\}, \{\neg y, u\}\} \quad R = \{\{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}\} \quad ((A \vee B) \wedge R)$$

Davis-Putnam Algorithm

Example (DP Algorithm)

$$\begin{aligned} F &= \{\{x, y, \neg z, u\}, \{\neg x, y, u\}, \{x, \neg y, \neg z\}, \{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}, \{\neg x, \neg y, u\}\} && \text{(Split by } x) \\ A &= \{\{y, \neg z, u\}, \{\neg y, \neg z\}\} \quad B = \{\{y, u\}, \{\neg y, u\}\} \quad R = \{\{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}\} && ((A \vee B) \wedge R) \\ F_1 &= \{\{y, \neg z, u\}, \{\neg y, \neg z, u\}, \{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}\} && \text{(Split by } y) \end{aligned}$$

Davis-Putnam Algorithm

Example (DP Algorithm)

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Davis-Putnam Algorithm

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Davis-Putnam Algorithm

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Davis-Putnam Algorithm

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Davis-Putnam Algorithm

Example (DP Algorithm)

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$$\begin{aligned} F &= \{\{x, y, \neg z, u\}, \{\neg x, y, u\}, \{x, \neg y, \neg z\}, \{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}, \{\neg x, \neg y, u\}\} && \text{(Split by } x) \\ A &= \{\{y, \neg z, u\}, \{\neg y, \neg z\}\} \quad B = \{\{y, u\}, \{\neg y, u\}\} \quad R = \{\{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}\} && ((A \vee B) \wedge R) \\ F_1 &= \{\{y, \neg z, u\}, \{\neg y, \neg z, u\}, \{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}\} && \text{(Split by } y) \\ A_1 &= \{\{\neg z, u\}\} \quad B_1 = \{\{\neg z, u\}\} \quad R_1 = \{\{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}\} && ((A_1 \vee B_1) \wedge R_1) \\ F_2 &= \{\{\neg z, u\}, \{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}\} && \text{(Split by } z) \\ A_2 &= \{\{v\}, \{\neg v\}\} \quad B_2 = \{\{u\}, \{\neg u\}\} \quad R_2 = \{\} && ((A_2 \vee B_2) \wedge R_2) \\ F_3 &= \{\{u, v\}, \{u, \neg v\}, \{\neg u, v\}, \{\neg u, \neg v\}\} && \text{(Split by } u) \\ A_3 &= \{\{v\}, \{\neg v\}\} \quad B_3 = \{\{v\}, \{\neg v\}\} \quad R_3 = \{\} && ((A_3 \vee B_3) \wedge R_3) \\ F_4 &= \{\{v\}, \{\neg v\}\} \vdash_1 \{\emptyset\} && \text{(Unit Resolution)} \end{aligned}$$

DP Variant: Bucket Elimination

Bucket Elimination

- Fix order \prec on variables.
- **Bucket:** set of clauses with same \prec -maximal variable
- **Bucket Elimination:** process buckets in decreasing \prec -order
 - resolve all clauses in bucket
 - put resolvents in fitting bucket

Example (Bucket Elimination)

$$F = \{(x, y, \bar{z}, u), (\bar{x}, y, u), (x, \bar{y}, \bar{z}), (z, v), (z, \bar{v}), (\bar{z}, \bar{u}), (\bar{x}, \bar{y}, u)\}$$

$$(x \succ y \succ z \succ u \succ v)$$

Variable	Bucket
x	$(x, y, \bar{z}, u), (\bar{x}, y, u), (x, \bar{y}, \bar{z}), (\bar{x}, \bar{y}, u)$
y	
z	$(z, v), (z, \bar{v}), (\bar{z}, \bar{u})$
u	
v	

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x	processed
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$$(x \succ y \succ z \succ u \succ v)$$

Variable	Bucket
x	processed
y	processed
z	processed
u	processed
v	$(v), (\bar{v})$

DP: Discussion

Excerpt from Davis' and Putnam's paper

*The superiority of the present procedure over those previously available is indicated in part by the fact that a formula on which **Gilmore's routine for the IBM 704** causes the machine to compute for **21 minutes without obtaining a result** was worked successfully by hand computation using [DP] in 30 minutes.*

- Does DP improve on saturation's average time complexity?

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- Does DP improve on saturation's average time complexity?
⇒ **yes** — if we split over the right variables
- Does DP avoid saturation's exponential space complexity?
⇒ **no** — **quadratic blowup in size** for eliminating one variable

DPLL Algorithm (Davis et al., 1962)

Davis Putnam Logemann Loveland (DPLL) Algorithm

- DPLL is a backtracking search over partial variable assignments.
- Case splitting over a variable x branches the search over two cases x and $\neg x$: resulting in the simplified formulas $F_{|x=\text{true}}$ and $F_{|x=\text{false}}$
- Simplification rules:
 - **Unit Propagation:** If $\{l\} \in F$, l must be set to true.
 - **Pure Literal Elimination:** If x occurs only positively (or only negatively), it may be fixed to the respective value.

DPLL Algorithm

Algorithm: DPLL(ClauseSet S)

```
1 while  $S$  contains a unit clause  $\{L\}$  do
2   | delete from  $S$  all clauses containing  $L$  // unit-subsumption
3   | delete  $\neg L$  from all clauses in  $S$  // unit-resolution
4 if  $\emptyset \in S$  then return false // empty clause
5 while  $S$  contains a pure literal  $L$  do
6   | delete from  $S$  all clauses containing  $L$  // pure literal elimination
7 if  $S = \emptyset$  then return true // no clauses
8 choose a literal  $L$  occurring in  $S$  // case-splitting
9 if  $DPLL(S \cup \{\{L\}\})$  then return true // first branch
10 else if  $DPLL(S \cup \{\{\neg L\}\})$  then return true // second branch
11 else return false
```

Start with
simplifications;

recurse on
subformulas obtained
by case-splitting;

stop if satisfying
assignment found or
all branches are
unsatisfiable.

DPLL Algorithm with Trail

Algorithm: trailDPLL(ClauseSet S , PartialAssignment α)

```
1 while ( $S, \alpha$ ) contains a unit clause  $\{L\}$  do
2 |   add  $\{L = 1\}$  to  $\alpha$                                 // Unit Propagation
3 if a literal is assigned both 0 and 1 in  $\alpha$  then
4 |   return false                                        // Conflict
5 if all literals assigned then
6 |   return true                                        // Assignment found
7 choose a literal  $L$  not assigned in  $\alpha$  occurring in  $S$  // Case Splitting
8 if trailDPLL( $S, \alpha \cup \{\{L = 1\}\}$ ) then
9 |   return true                                        // first branch
10 else if trailDPLL( $S, \alpha \cup \{\{L = 0\}\}$ ) then
11 |   return true                                        // second branch
12 else return false
```

(S, α) is the clause set S
as “seen” under partial
assignment α

No explicit pure literal
elimination (too slow for the
benefit it provides)

trailDPLL() leads to efficient
iterative DPLL
implementation

DPLL Algorithm

Properties

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DPLL Algorithm

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- Space complexity:

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 - If clause set S is SAT, we eventually find a satisfying α
 - If clause set S is UNSAT, the entire space of (partial) variable assignments is searched (but variable selection still matters!)
- Space complexity: linear!
 - Systematic search avoids blowup of “unfocused” DP

Recap

Elementary Algorithms

- Local Search
 - Examples: GSAT, WalkSAT
 - Terminology: breakcount, makecount, diffscore
- Resolution
 - Soundness and Completeness
 - Saturation Algorithm (Exponential Complexity)
- DP Algorithm
 - Systematized Resolution
 - Improved Average Time Complexity
- DPLL Algorithm
 - Case Splitting and Unit Propagation
 - Linear Space Complexity